Characterizing Independence with Tensor Product Kernels

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- Kullback-Leibler divergence:

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Alternatives: Rényi, Tsallis, L^2 divergence... Typically: $\mathcal{X} = \mathbb{R}^d$.

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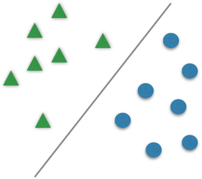

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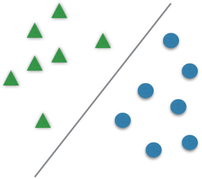

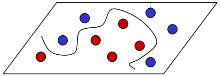

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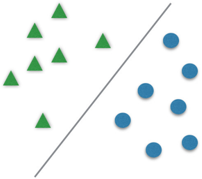

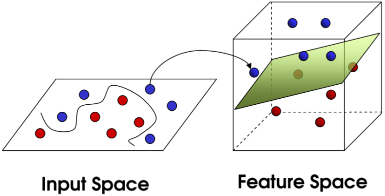

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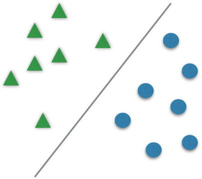
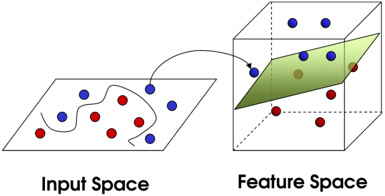


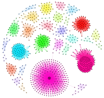


2. Representation of distributions:

$$\mathbb{P} \mapsto \mathbb{E}_{\mathbf{x}\sim\mathbb{P}}\varphi(\mathbf{x}).$$

$\varphi(\mathbf{x}) = \mathbf{x}$: mean, $\varphi(\mathbf{x}) = e^{i\langle\cdot,\mathbf{x}\rangle}$: characteristic function.

Diverse Set of Domains, Kernel Examples



- $\mathcal{X} = \mathbb{R}^d$, $\gamma > 0$:

$$k_p(\mathbf{x}, \mathbf{y}) = (\langle \mathbf{x}, \mathbf{y} \rangle + \gamma)^p, \qquad k_G(\mathbf{x}, \mathbf{y}) = e^{-\gamma \|\mathbf{x}-\mathbf{y}\|_2^2},$$

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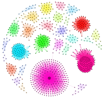

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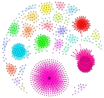

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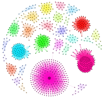

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- $\mathcal{X}$ = trees, graphs, dynamical systems, sets, permutations, ...

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'KL divergence & mutual information' on kernel-endowed domains.

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When is HSIC an independence measure? Conditions on k_m-s?

Ingredients

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Assumption

$\mathcal{X}_m$: kernel-enriched domains.

Ingredients: Kernel, RKHS ($\mathcal{X} := \mathcal{X}_m$, $k := k_m$)

Given: $\mathcal{X}$ set. $\mathcal{H}$(ilbert space).

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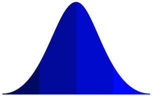

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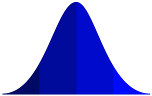

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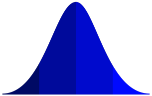

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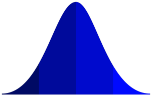

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Equivalent definitions. We represent distributions in an RKHS...

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Mean Embedding, MMD: Applications & Review

- Applications:
  - two-sample testing [Gretton et al., 2012], domain adaptation [Zhang et al., 2013], -generalization [Blanchard et al., 2017],
  - interpretable machine learning [Kim et al., 2016],
  - kernel belief propagation [Song et al., 2011], kernel Bayes' rule [Fukumizu et al., 2013], model criticism [Lloyd et al., 2014],
  - approximate Bayesian computation [Park et al., 2016], probabilistic programming [Schölkopf et al., 2015],
  - distribution classification [Muandet et al., 2011], distribution regression [Szabó et al., 2016], topological data analysis [Kusano et al., 2016].
- Review [Muandet et al., 2017].

Let us switch to HSIC.

MMD $\xrightarrow{\text{spec.}}$ HSIC

MMD with $k = \otimes_{m=1}^{M} k_m$:

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Applications:

- blind source separation [Gretton et al., 2005],
- feature selection [Song et al., 2012], post selection inference [Yamada et al., 2016],
- independence testing [Gretton et al., 2008], causal inference [Mooij et al., 2016, Pfister et al., 2017, Strobl et al., 2017].

Central in Applications: Characteristic Property

- MMD: k is called characteristic [Fukumizu et al., 2008] if

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Wanted

- $\otimes_{m=1}^M k_m$ is $\mathcal{I}$-characteristic: conditions in terms of k_m-s?
- $\otimes_{m=1}^M k_m$ is characteristic: relation?

Characteristic Property: Description on $\mathbb{R}^d$

For continuous bounded shift-invariant kernels on $\mathbb{R}^d$:

$$k(\mathbf{x},\mathbf{x}') = k_0(\mathbf{x}-\mathbf{x}') \overset{(*)}{=} \int_{\mathbb{R}^d} e^{-i\langle \mathbf{x}-\mathbf{x}',\boldsymbol{\omega}\rangle} \mathrm{d}\Lambda(\boldsymbol{\omega})$$

$(*)$: Bochner's theorem.

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$$\|\mu_{\mathbb{P}}-\mu_{\mathbb{Q}}\|_{\mathcal{H}_k} = \|c_{\mathbb{P}}-c_{\mathbb{Q}}\|_{L^2(\Lambda)}.$$

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Theorem ([Sriperumbudur et al., 2010])

k is characteristic iff. $supp(\Lambda) = \mathbb{R}^d$.

Examples on $\mathbb{R}$; Similarly $\mathbb{R}^d$

kernel name	k_0	$\widehat{k_0}(\omega)$	$supp\left(\widehat{k_0}\right)$
Gaussian	$e^{-\frac{x^2}{2\sigma^2}}$	$\sigma e^{-\frac{\sigma^2\omega^2}{2}}$	$\mathbb{R}$
Laplacian	$e^{-\sigma\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{\sigma}{\sigma^2+\omega^2}$	$\mathbb{R}$
B_{2n+1}-spline	$*^{2n+2}\chi_{\left[-\frac{1}{2},\frac{1}{2}\right]}(x)$	$\frac{4^{n+1}}{\sqrt{2\pi}}\frac{\sin^{2n+2}\left(\frac{\omega}{2}\right)}{\omega^{2n+2}}$	$\mathbb{R}$
Sinc	$\frac{\sin(\sigma x)}{x}$	$\sqrt{\frac{\pi}{2}}\chi_{[-\sigma,\sigma]}(\omega)$	$[-\sigma,\sigma]$
Fejér	$\frac{1}{n+1}\frac{\sin^2\frac{(n+1)x}{2}}{\sin^2\left(\frac{x}{2}\right)}$	$\sqrt{2\pi}\sum_{j=-n}^{n}\left(1-\frac{\lvert j\rvert}{n+1}\right)\delta(\omega-j)$	$\{0,\pm1,\pm2,\dots,\pm n\}$

Universality [Steinwart, 2001, Sriperumbudur et al., 2010]

k is called characteristic (recall!)/universal if

$$\mu_k : \underbrace{\mathcal{M}_1^+(\mathcal{X})}_{\text{probability measures on } \mathcal{X}} \mapsto \mathcal{H}_k, \qquad \mu_k : \underbrace{\mathcal{M}_b(\mathcal{X})}_{\text{bounded signed measures on } \mathcal{X}} \mapsto \mathcal{H}_k$$

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Challenge

Characteristic/$\mathcal{I}$-characteristic/universality of $\otimes_{m=1}^M k_m$ in terms of k_m-s!

Existing Results, $M = 2$

- [Blanchard et al., 2011, Waegeman et al., 2012, Gretton, 2015]:
  $k_1 \& k_2$: universal $\Rightarrow k_1 \otimes k_2$: universal ($\Rightarrow \mathcal{I}$-characteristic).

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Extension to $M \geqslant 2$.

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Goal

Extension to $M \geqslant 2$.

Main Challenge

'$\otimes k_m$: $\mathcal{I}$-characteristic $\Leftrightarrow k_m$: characteristic $(\forall m)$' does NOT hold.

Idea: Characteristic Property as Ispd

- Characteristic property:

$$\mathbb{F} = \mathbb{P}_1 - \mathbb{P}_2 \neq 0 \Rightarrow \mu_{\mathbb{F}} \neq 0.$$

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- Observation [Sriperumbudur et al., 2010]: k is characteristic iff.

$$\|\mu_{\mathbb{F}}\|^2_{\mathcal{H}_k} > 0, \ \forall \underbrace{\mathbb{F} \in \mathcal{M}_b(\mathcal{X}) \backslash \{0\} \ \ \mathbb{F}(\mathcal{X}) = 0}_{\mathcal{F}_1}.$$

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$$\underbrace{\|\mu_{\mathbb{F}}\|^2_{\mathcal{H}_k}}_{\int_{\mathcal{X}}\int_{\mathcal{X}} k(x,x')\,\mathrm{d}\mathbb{F}(x)\,\mathrm{d}\mathbb{F}(x')} > 0, \ \forall \underbrace{\mathbb{F} \in \mathcal{M}_b(\mathcal{X})\backslash\{0\} \quad \mathbb{F}(\mathcal{X}) = 0}_{\mathcal{F}_1}.$$

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- We saw: k is universal iff.

$$\|\mu_{\mathbb{F}}\|^2_{\mathcal{H}_k} > 0, \ \forall \underbrace{\mathbb{F} \in \mathcal{M}_b(\mathcal{X})\backslash\{0\}}_{\mathcal{F}_2}.$$

$\mathcal{F}$-ispd Tensor Product Kernels

From now on: $\mathcal{X} = \times_{m=1}^{M} \mathcal{X}_m$. Let $\mathcal{F} \subseteq \mathcal{M}_b(\mathcal{X})$, $0 \in \mathcal{F}$.

Definition

$k = \otimes_{m=1}^{M} k_m$ is called $\mathcal{F}$-ispd if

$$\|\mu_k(\mathbb{F})\|_{\mathcal{H}_k}^2 > 0, \quad \forall\, \mathbb{F} \in \mathcal{F} \backslash \{0\}$$

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$$\|\mu_k(\mathbb{F})\|_{\mathcal{H}_k}^2 > 0, \quad \forall\, \mathbb{F} \in \mathcal{F} \backslash \{0\}, \text{ equivalently}$$

$$\mu_k(\mathbb{F}) = 0 \Rightarrow \mathbb{F} = 0 \quad (\mathbb{F} \in \mathcal{F}).$$

Examples

$\mathcal{F}$	$\mathcal{F}$-ispd k
$\mathcal{M}_b(\mathcal{X})$	universal
$[\mathcal{M}_b(\mathcal{X})]^0$	characteristic

$$\subseteq \quad \subseteq \quad [\mathcal{M}_b(\mathcal{X})]^0 \subseteq \mathcal{M}_b(\mathcal{X}) .$$

$$\cup\!\!\mid$$

$\Leftarrow$ $\Leftarrow$ characteristic $\Leftarrow$ universal.

$$\Downarrow$$

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$\mathcal{I} := \{\mathbb{P} - \otimes_{m=1}^M \mathbb{P}_m\}$	$\mathcal{I}$-characteristic

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$[\mathcal{M}_b(\mathcal{X})]^0$	characteristic
$\mathcal{I} := \{\mathbb{P} - \otimes_{m=1}^M \mathbb{P}_m\}$	$\mathcal{I}$-characteristic
$[\otimes_{m=1}^M \mathcal{M}_b(\mathcal{X}_m)]^0$	$\otimes$-characteristic

$$\subseteq \left[\otimes_{m=1}^M \mathcal{M}_b(\mathcal{X}_m)\right]^0 \subseteq \left[\mathcal{M}_b(\mathcal{X})\right]^0 \subseteq \mathcal{M}_b(\mathcal{X}) .$$

$$\left[\mathcal{M}_b(\mathcal{X})\right]^0 \supseteq \mathcal{I}$$

$\Leftarrow$ $\otimes$ -characteristic $\Leftarrow$ characteristic $\Leftarrow$ universal.

characteristic $\Downarrow$ $\mathcal{I}$-characteristic

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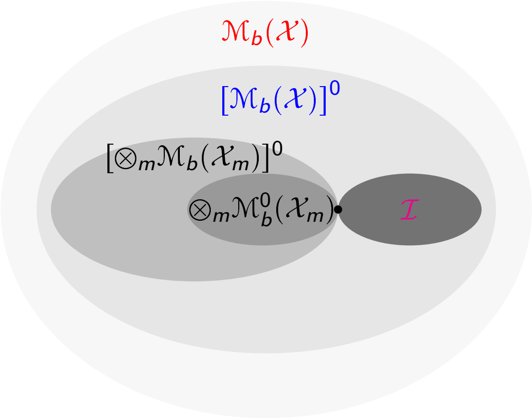
$\mathcal{M}_b(\mathcal{X})$	universal
$[\mathcal{M}_b(\mathcal{X})]^0$	characteristic
$\mathcal{I} := \{\mathbb{P} - \otimes_{m=1}^M \mathbb{P}_m\}$	$\mathcal{I}$-characteristic
$[\otimes_{m=1}^M \mathcal{M}_b(\mathcal{X}_m)]^0$	$\otimes$-characteristic
$\otimes_{m=1}^M \mathcal{M}_b^0(\mathcal{X}_m)$	$\otimes_0$-characteristic

$$\otimes_{m=1}^M \mathcal{M}_b^0(\mathcal{X}_m) \subseteq [\otimes_{m=1}^M \mathcal{M}_b(\mathcal{X}_m)]^0 \subseteq [\mathcal{M}_b(\mathcal{X})]^0 \subseteq \mathcal{M}_b(\mathcal{X}).$$

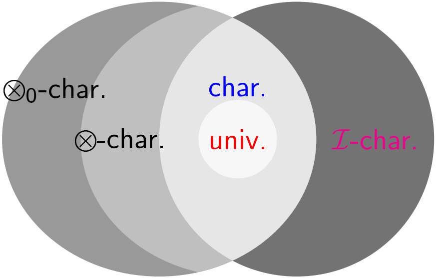
$$[\mathcal{M}_b(\mathcal{X})]^0 \supseteq \mathcal{I}$$

$\otimes_0$-characteristic $\Leftarrow$ $\otimes$-characteristic $\Leftarrow$ characteristic $\Leftarrow$ universal.

characteristic $\Downarrow$ $\mathcal{I}$-characteristic



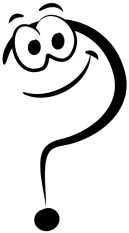
$\mathcal{M}_b(\mathcal{X})$
$[\mathcal{M}_b(\mathcal{X})]^0$
$[\otimes_m \mathcal{M}_b(\mathcal{X}_m)]^0$
$\otimes_m \mathcal{M}_b^0(\mathcal{X}_m)$
$\mathcal{I}$



$\otimes_0$-char.
char.
$\otimes$-char.
univ.
$\mathcal{I}$-char.

$$\otimes_0\text{-char} \Longleftarrow \otimes\text{-char} \Longleftarrow \text{char} \Longleftarrow \text{universal}$$

$$\text{char} \Longrightarrow \mathcal{I}\text{-char}$$



$$(k_m)_{m=1}^M \text{ char} \;\overset{\text{[Sriperumbudur et al., 2011]}}{\not\Longrightarrow}\; (k_m)_{m=1}^M\text{-universal}$$

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Results

Various Characteristic Properties of $\otimes_{m=1}^M k_m$

Proposition

(i) $\otimes_{m=1}^M k_m$: *characteristic* $\Rightarrow$ $\otimes$*-characteristic.*

(ii) $\otimes_{m=1}^M k_m$: $\otimes$*-characteristic* $\Rightarrow$ $\otimes_0$*-characteristic.*

(iii) $\otimes_{m=1}^M k_m$: $\otimes_0$*-characteristic* $\Leftrightarrow$ $(k_m)_{m=1}^M$ *are characteristic.*

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(iii) $\otimes_{m=1}^M k_m$: $\otimes_0$-*characteristic* $\Leftrightarrow$ $(k_m)_{m=1}^M$ *are characteristic.*

(iii) remains. Idea: with $k = \otimes_{m=1}^M k_m$, $\mathbb{F} = \otimes_{m=1}^M \mathbb{F}_m$,

$$\underbrace{\|\mu_k(\mathbb{F})\|_{\mathcal{H}_k}^2}_{>0} = \underbrace{\prod_{m=1}^M}_{\forall} \underbrace{\|\mu_{k_m}(\mathbb{F}_m)\|_{\mathcal{H}_{k_m}}^2}_{>0},$$

$\otimes_0$-characteristic $\not\Rightarrow$ even $\otimes$-characteristic

Reverse of (ii) does not hold.

Example

- $\mathcal{X}_m = \{1,2\}$, $\tau_{\mathcal{X}_m} = \mathcal{P}(\{1,2\})$, $k_m(x,x') = 2\delta_{x,x'} - 1$, $M = 2$.
- $k_1 = k_2$: *characteristic, but* $k_1 \otimes k_2$ *is not* $\otimes$*-characteristic.*
- $k_1 \otimes k_2$ *is* $\mathcal{I}$*-characteristic.*

Proof Idea: $k_1 \otimes k_2$: not $\otimes$-characteristic

Finite signed measures on $\mathcal{X}_m = \{1, 2\}$:

$$\mathbb{F}_1(\mathbf{a}) = a_1\delta_1 + a_2\delta_2, \qquad \mathbb{F}_2(\mathbf{b}) = b_1\delta_1 + b_2\delta_2.$$

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Goal: construct a witness $0 \neq \mathbb{F} = \mathbb{F}_1 \otimes \mathbb{F}_2 \in \otimes_{m=1}^2 \mathcal{M}_b(\mathcal{X}_m)$ s.t.

$$0 = \mathbb{F}(\mathcal{X}_1 \times \mathcal{X}_2) = \mathbb{F}_1(\mathcal{X}_1)\mathbb{F}_2(\mathcal{X}_2),$$

$$0 = \int_{\mathcal{X}_1 \times \mathcal{X}_2} \int_{\mathcal{X}_1 \times \mathcal{X}_2} k_1(x_1, x_1')k_2(x_2, x_2')\, \mathrm{d}\mathbb{F}(x_1, x_2)\, \mathrm{d}\mathbb{F}(x_1', x_2').$$

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This gives

$$0 = (a_1 + a_2)(b_1 + b_2), \qquad 0 = (a_1 - a_2)^2(b_1 - b_2)^2.$$

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This gives

$$0 = (a_1 + a_2)(b_1 + b_2), \qquad 0 = (a_1 - a_2)^2(b_1 - b_2)^2.$$

$\Rightarrow$ Two symmetric solutions ($\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$):

$$a_1 + a_2 = 0, \qquad b_1 = b_2.$$

$$a_1 = a_2, \qquad b_1 + b_2 = 0.$$

Towards $\mathcal{I}$-characteristicity

In the previous example:

$$k_1, k_2\text{: characteristic} \Rightarrow k_1 \otimes k_2\text{: } \mathcal{I}\text{-characteristic.}$$

In fact:

- this holds for any bounded kernel,
- +converse for any $M \geqslant 2$! Formally, ...

$\mathcal{I}$-characteristic Property

Proposition

(i) k_1, k_2: *characteristic* $\Rightarrow$ $k_1 \otimes k_2$: $\mathcal{I}$-*characteristic.*

(ii) $\otimes_{m=1}^{M} k_m$: $\mathcal{I}$-*characteristic* $\Rightarrow$ $(k_m)_{m=1}^{M}$ *are characteristic.*

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Proof idea:

(i) Induction: see later universality $(\mathbb{F} = \mathbb{P} - \mathbb{P}_1 \otimes \mathbb{P}_2)$.

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Proof idea:

(i) Induction: see later universality ($\mathbb{F} = \mathbb{P} - \mathbb{P}_1 \otimes \mathbb{P}_2$).

(ii) If a k_m is not characteristic, witness can be constructed.

Example

- $\mathcal{X}_m = \{1,2\}$, $\tau_{\mathcal{X}_m} = \mathcal{P}(\{1,2\})$, $k_m(x,x') = 2\delta_{x,x'} - 1$, $M = 3$.
- *Then*
  - $(k_m)_{m=1}^3$: *characteristic.*
  - $\otimes_{m=1}^3 k_m$: *is not $\mathcal{I}$-characteristic. Witness:*

$$p_{1,1,1} = \frac{1}{5}, \quad p_{1,1,2} = \frac{1}{10}, \quad p_{1,2,1} = \frac{1}{10}, \quad p_{1,2,2} = \frac{1}{10},$$
$$p_{2,1,1} = \frac{1}{5}, \quad p_{2,1,2} = \frac{1}{10}, \quad p_{2,2,1} = \frac{1}{10}, \quad p_{2,2,2} = \frac{1}{10}.$$

Non-$\mathcal{I}$-characteristicity: Analytical Solution

Parameter: $\mathbf{z} = (z_0, z_1, \ldots, z_5) \in [0,1]^6$.

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$$-\frac{\begin{aligned}&z_2 + z_1 + z_4 + z_5 - 3z_2z_1 - 4z_2z_4 - 4z_1z_4 - z_2z_3 - 2z_2z_0 - 2z_1z_3 - 3z_2z_5\\&- 2z_4z_3 - z_1z_0 - 3z_1z_5 - 2z_4z_0 - 4z_4z_5 - z_3z_0 - z_3z_5 - z_0z_5 + 2z_2z_1^2 + 2z_2^2z_1\\&+ 4z_2z_4^2 + 2z_2^2z_4 + 4z_1z_4^2 + 2z_1^2z_4 + 2z_2^2z_0 + 2z_1^2z_3 + 2z_2z_5^2 + 2z_2^2z_5 + 2z_4^2z_3\\&+ 2z_1z_5^2 + 2z_1^2z_5 + 2z_4^2z_0 + 2z_4z_5^2 + 4z_4^2z_5 - z_2^2 - z_1^2 - 3z_4^2 + 2z_4^3 - z_5^2\\&+ 6z_2z_1z_4 + 2z_2z_1z_3 + 2z_2z_4z_3 + 2z_2z_1z_0 + 4z_2z_1z_5 + 4z_2z_4z_0 + 4z_1z_4z_3\\&+ 6z_2z_4z_5 + 2z_1z_4z_0 + 6z_1z_4z_5 + 2z_2z_3z_0 + 2z_2z_3z_5 + 2z_1z_3z_0 + 2z_2z_0z_5\\&+ 2z_1z_3z_5 + 2z_4z_3z_0 + 2z_4z_3z_5 + 2z_1z_0z_5 + 2z_4z_0z_5\end{aligned}}{\begin{aligned}&2z_2z_1 - z_1 - 2z_4 - z_3 - z_0 - 2z_5 - z_2 + 2z_2z_4 + 2z_1z_4 + 2z_2z_0 + 2z_1z_3 + 2z_2z_5\\&+ 2z_4z_3 + 2z_1z_5 + 2z_4z_0 + 4z_4z_5 + 2z_3z_0 + 2z_3z_5 + 2z_0z_5 + 2z_4^2 + 2z_5^2\end{aligned}}.$$

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$$
-\frac{\begin{array}{l} z_2 + z_1 + z_4 + z_5 - 3z_2z_1 - 4z_2z_4 - 4z_1z_4 - z_2z_3 - 2z_2z_0 - 2z_1z_3 - 3z_2z_5 \\ - 2z_4z_3 - z_1z_0 - 3z_1z_5 - 2z_4z_0 - 4z_4z_5 - z_3z_0 - z_3z_5 - z_0z_5 + 2z_2z_1^2 + 2z_2^2z_1 \\ + 4z_2z_4^2 + 2z_2^2z_4 + 4z_1z_4^2 + 2z_1^2z_4 + 2z_2^2z_0 + 2z_1^2z_3 + 2z_2z_5^2 + 2z_2^2z_5 + 2z_4^2z_3 \\ + 2z_1z_5^2 + 2z_1^2z_5 + 2z_4^2z_0 + 2z_4z_5^2 + 4z_4^2z_5 - z_2^2 - z_1^2 - 3z_4^2 + 2z_4^3 - z_5^2 \\ + 6z_2z_1z_4 + 2z_2z_1z_3 + 2z_2z_4z_3 + 2z_2z_1z_0 + 4z_2z_1z_5 + 4z_2z_4z_0 + 4z_1z_4z_3 \\ + 6z_2z_4z_5 + 2z_1z_4z_0 + 6z_1z_4z_5 + 2z_2z_3z_0 + 2z_2z_3z_5 + 2z_1z_3z_0 + 2z_2z_0z_5 \\ + 2z_1z_3z_5 + 2z_4z_3z_0 + 2z_4z_3z_5 + 2z_1z_0z_5 + 2z_4z_0z_5 \end{array}}{\begin{array}{l} 2z_2z_1 - z_1 - 2z_4 - z_3 - z_0 - 2z_5 - z_2 + 2z_2z_4 + 2z_1z_4 + 2z_2z_0 + 2z_1z_3 + 2z_2z_5 \\ + 2z_4z_3 + 2z_1z_5 + 2z_4z_0 + 4z_4z_5 + 2z_3z_0 + 2z_3z_5 + 2z_0z_5 + 2z_4^2 + 2z_5^2 \end{array}}.
$$

We chose: $\mathbf{z} = \left(\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right)$.

$\mathbb{R}^d$ & Translation-invariance: All Notions Coincide

Proposition

Assume $k_m : \mathbb{R}^{d_m} \times \mathbb{R}^{d_m} \to \mathbb{R}$ are continuous, translation-invariant kernels. Then the followings are equivalent:

(i) $(k_m)_{m=1}^M$*-s are characteristic.*

(ii) $\otimes_{m=1}^M k_m$*:* $\otimes_0$*-characteristic.*

(iii) $\otimes_{m=1}^M k_m$*:* $\otimes$*-characteristic.*

(iv) $\otimes_{m=1}^M k_m$*:* $\mathcal{I}$*-characteristic.*

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Proof idea: We already know

$$(v) \Rightarrow (iii) \Rightarrow (ii) \Leftrightarrow (i), \qquad (v) \Rightarrow (iv) \Rightarrow (i).$$

Remains: $(i) \Rightarrow (v)$.

- Since k_m is characteristic

$$k_m \xrightarrow{\text{Bochner thm}} \Lambda_m, \; supp(\Lambda_m) = \mathbb{R}^{d_m}.$$

$(k_m)_{m=1}^M$: characteristic $\Rightarrow \otimes_{m=1}^M k_m$: characteristic

- Since k_m is characteristic

$$k_m \xrightarrow{\text{Bochner thm}} \Lambda_m, \; supp(\Lambda_m) = \mathbb{R}^{d_m}.$$

- Tensor kernel:

$$\otimes_{m=1}^M k_m \xrightarrow{\text{Bochner thm}} \Lambda = \otimes_{m=1}^M \Lambda_m.$$

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- Tensor kernel:

$$\otimes_{m=1}^M k_m \xrightarrow{\text{Bochner thm}} \Lambda = \otimes_{m=1}^M \Lambda_m.$$

- $\text{supp}(\Lambda) = \times_{m=1}^M \underbrace{\text{supp}\left(\Lambda_m\right)}_{\mathbb{R}^{d_m}} = \mathbb{R}^d.$

Universality of $\otimes_{m=1}^{M} k_m$

We saw: for $M \geqslant 3$

$$(k_m)_{m=1}^{M} \text{ are characteristic} \not\Rightarrow \otimes_{m=1}^{M} k_m\text{: } \mathcal{I}\text{-characteristic.}$$

Proposition

$$\otimes_{m=1}^{M} k_m\text{: universal} \Leftrightarrow (k_m)_{m=1}^{M} \text{ are universal.}$$

The Tricky Direction: If $(k_m)_{m=1}^M$ are Universal . . .

Goal: injectivity of $\mu = \mu_{\otimes_{m=1}^M k_m}$ on $\mathcal{M}_b(\mathcal{X})$, i.e.

$$\mu(\mathbb{F}) = 0 \overset{?}{\Rightarrow} \mathbb{F} = 0.$$

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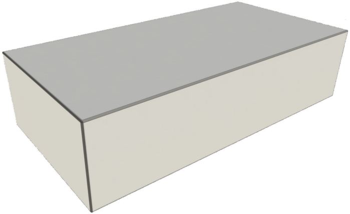
Enough:

$$\mathbb{F}\left(\times_{m=1}^M B_m\right) = 0, \quad \forall B_m.$$



Proof Idea

$$0 = \mu(\mathbb{F}) = \int_{\mathcal{X}} \otimes_{m=1}^{M} k_m(\cdot, x_m) \mathrm{d}\mathbb{F}(x),$$

$$0 = \mathbb{F}\left(\times_{m=1}^{M} B_m\right) = \int_{\mathcal{X}} \times_{m=1}^{M} \chi_{B_m}(x_m) \mathrm{d}\mathbb{F}(x), \ \forall B_m.$$

Proof Idea

$$0 = \mu(\mathbb{F}) = \int_{\mathcal{X}} \otimes_{m=1}^{M} k_m(\cdot, x_m) \mathrm{d}\mathbb{F}(x),$$

$$0 = \int_{\mathcal{X}} \prod_{m=1}^{J} \chi_{B_m}(x_m) \otimes_{m=J+1}^{M} k_m(\cdot, x_m) \, \mathrm{d}\mathbb{F}(x), \ \forall B_m,$$

$$0 = \mathbb{F}\left(\times_{m=1}^{M} B_m\right) = \int_{\mathcal{X}} \times_{m=1}^{M} \chi_{B_m}(x_m) \mathrm{d}\mathbb{F}(x), \ \forall B_m.$$

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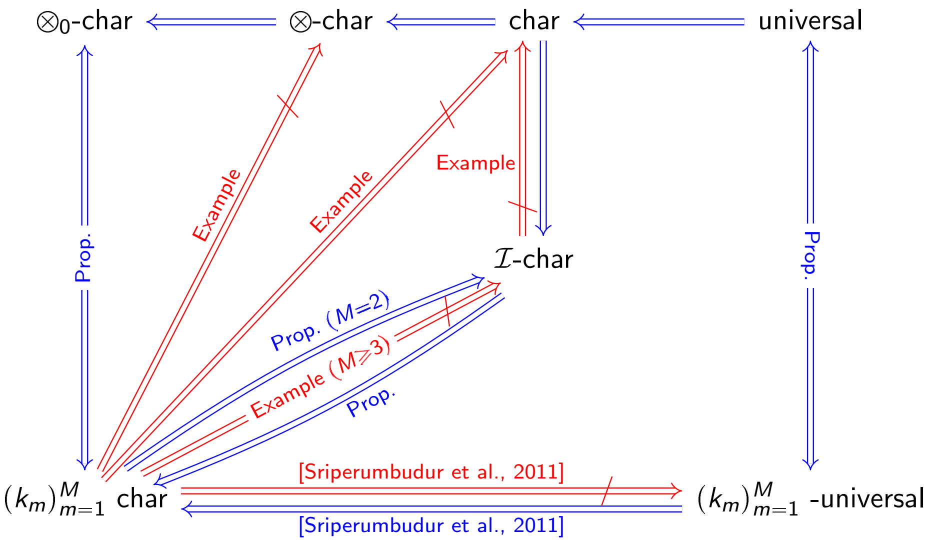
We proceed by induction ($J = 0, \ldots, M$).



⊗0-char
⊗-char
char
universal
Example
Example
Example
I-char
Prop.
Prop.
Prop. (M=2)
Example (M⩾3)
Prop.
[Sriperumbudur et al., 2011]
(k_m)_{m=1}^M char
(k_m)_{m=1}^M -universal
[Sriperumbudur et al., 2011]

Summary

We studied the validness of HSIC.

- HSIC $\Rightarrow$ product structure:
  - Space: $\mathcal{X} = \times_{m=1}^{M} \mathcal{X}_m$.
  - Kernel: $k = \otimes_{m=1}^{M} k_m$.
- $\mathcal{F}$-ispd property $\Rightarrow$ complete answer in terms of k_m-s.

Summary

We studied the validness of HSIC.

- HSIC ⇒ product structure:
  - Space: $\mathcal{X} = \times_{m=1}^{M} \mathcal{X}_m$.
  - Kernel: $k = \otimes_{m=1}^{M} k_m$.
- $\mathcal{F}$-ispd property ⇒ complete answer in terms of k_m-s.
- ITE toolkit, preprint (maths → JMLR):

https://bitbucket.org/szzoli/ite/

http://arxiv.org/abs/1708.08157

Thank you for the attention!

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