Towards Large-Scale Approximation Of Tasks With Derivatives - A Kernel Perspective

Zoltán Szabó – CMAP, École Polytechnique

Joint work with:

- Linda Chamakh @ CMAP & BNP Paribas
- Emmanuel Gobet @ CMAP

M3HPCST,
Ghaziabad, India
January 9, 2020

- Def-1 (feature space):

$$k(x, y) = \langle \varphi(x), \varphi(y) \rangle_{\mathcal{H}} .$$

Kernel, RKHS: generalized inner product, -linear methods

- Def-1 (feature space):

$$k(x, y) = \langle \varphi(x), \varphi(y) \rangle_{\mathcal{H}}.$$

- Def-2 (reproducing kernel):

$$k(\cdot, x) \in \mathcal{H}, \qquad f(x) = \langle f, k(\cdot, x) \rangle_{\mathcal{H}}.$$

Constructively, $\mathcal{H}_k = \overline{\{\sum_{i=1}^{n} \alpha_i k(\cdot, x_i)\}}$.

Kernel, RKHS: generalized inner product, -linear methods

- Def-1 (feature space):

$$k(x,y) = \langle \varphi(x), \varphi(y) \rangle_{\mathcal{H}}.$$

- Def-2 (reproducing kernel):

$$k(\cdot, x) \in \mathcal{H}, \qquad f(x) = \langle f, k(\cdot, x) \rangle_{\mathcal{H}}.$$

Constructively, $\mathcal{H}_k = \overline{\{\sum_{i=1}^n \alpha_i k(\cdot, x_i)\}}$.

- Def-3 (Gram matrix): $\mathbf{G} = [k(x_i, x_j)]_{i,j=1}^n \in \mathbb{R}^{n \times n} \succeq 0$.

Kernel, RKHS: generalized inner product, -linear methods

- Def-1 (feature space):

$$k(x,y) = \langle \varphi(x), \varphi(y) \rangle_{\mathcal{H}}.$$

- Def-2 (reproducing kernel):

$$k(\cdot, x) \in \mathcal{H}, \qquad f(x) = \langle f, k(\cdot, x) \rangle_{\mathcal{H}}.$$

  Constructively, $\mathcal{H}_k = \overline{\{\sum_{i=1}^{n} \alpha_i k(\cdot, x_i)\}}$.
- Def-3 (Gram matrix): $\mathbf{G} = [k(x_i, x_j)]_{i,j=1}^{n} \in \mathbb{R}^{n \times n} \succeq 0$.
- Def-4 (evaluation): $\delta_x(f) = f(x)$ is continuous for all x.

Kernel, RKHS: generalized inner product, -linear methods

- Def-1 (feature space):

$$k(x,y) = \langle \varphi(x), \varphi(y) \rangle_{\mathcal{H}}.$$

- Def-2 (reproducing kernel):

$$k(\cdot, x) \in \mathcal{H}, \qquad f(x) = \langle f, k(\cdot, x) \rangle_{\mathcal{H}}.$$

Constructively, $\mathcal{H}_k = \overline{\{\sum_{i=1}^n \alpha_i k(\cdot, x_i)\}}$.

- Def-3 (Gram matrix): $\mathbf{G} = [k(x_i, x_j)]_{i,j=1}^n \in \mathbb{R}^{n \times n} \succeq 0$.
- Def-4 (evaluation): $\delta_x(f) = f(x)$ is continuous for all x.

- All these definitions are equivalent, $k \overset{1:1}{\leftrightarrow} \mathcal{H}_k$.

Kernel examples: $\gamma > 0$, $p \in \mathbb{Z}^+$

$$k_p(\mathbf{x},\mathbf{y}) = (\langle \mathbf{x},\mathbf{y}\rangle + \gamma)^p, \qquad k_G(\mathbf{x},\mathbf{y}) = e^{-\gamma\|\mathbf{x}-\mathbf{y}\|_2^2},$$

$$k_e(\mathbf{x},\mathbf{y}) = e^{-\gamma\|\mathbf{x}-\mathbf{y}\|_2}, \qquad k_C(\mathbf{x},\mathbf{y}) = \frac{1}{1+\gamma\|\mathbf{x}-\mathbf{y}\|_2^2},$$

$$k_L(\mathbf{x},\mathbf{y}) = e^{-\gamma\|\mathbf{x}-\mathbf{y}\|_1}, \qquad \ldots$$

Kernel examples: $\gamma > 0$, $p \in \mathbb{Z}^+$

$$k_p(\mathbf{x},\mathbf{y}) = (\langle \mathbf{x},\mathbf{y}\rangle + \gamma)^p, \qquad k_G(\mathbf{x},\mathbf{y}) = e^{-\gamma \left\| \boxed{\mathbf{x}-\mathbf{y}} \right\|_2^2},$$

$$k_e(\mathbf{x},\mathbf{y}) = e^{-\gamma \left\| \boxed{\mathbf{x}-\mathbf{y}} \right\|_2}, \qquad k_C(\mathbf{x},\mathbf{y}) = \frac{1}{1+\gamma \left\| \boxed{\mathbf{x}-\mathbf{y}} \right\|_2^2},$$

$$k_L(\mathbf{x},\mathbf{y}) = e^{-\gamma \left\| \boxed{\mathbf{x}-\mathbf{y}} \right\|_1}, \qquad \ldots$$

Today

- $\mathcal{X} = \mathbb{R}^d$,
- continuous, bounded, shift-invariant k.

Empirical risk minimization

Classical problem

$$\min_{f \in \mathcal{H}_k} C(f) := \frac{1}{N} \sum_{n \in [N]} L(f(\mathbf{x}_i), y_i) + \lambda \|f\|_{\mathcal{H}_k}^2 \quad (\lambda > 0).$$

Examples:

- $L(a, b) = (a - b)^2$: kernel ridge regression.
- $L(a, b) = |a - b|_\epsilon$: ϵ-insensitive regression.
- $L(a, b) = \max(1 - ab, 0)$: classification using hinge loss.

ERM with derivatives

In fact, often the task:

$$\min_{f\in\mathcal{H}_k} C\left(\left\{\partial^{\mathbf{p}} f(\mathbf{x}_n)\right\}_{\substack{n\in[N]\\ \mathbf{p}\in D_n}}, \|f\|_{\mathcal{H}_k}^2\right) \quad \partial^{\mathbf{p}} f(\mathbf{x}_n) := \frac{\partial^{p_1+\ldots+p_d} f(\mathbf{x}_n)}{\partial_{x_1}^{p_1}\cdots\partial_{x_d}^{p_d}}.$$

ERM with derivatives

In fact, often the task:

$$\min_{f\in\mathcal{H}_k} C\left(\left\{\partial^{\mathbf{p}} f(\mathbf{x}_n)\right\}_{\substack{n\in[N]\\ \mathbf{p}\in D_n}}, \|f\|_{\mathcal{H}_k}^2\right) \quad \partial^{\mathbf{p}} f(\mathbf{x}_n) := \frac{\partial^{p_1+\ldots+p_d} f(\mathbf{x}_n)}{\partial_{x_1}^{p_1}\cdots\partial_{x_d}^{p_d}}.$$

Examples: semi-supervised learning with gradient information [Zhou, 2008], nonlinear variable selection [Rosasco et al., 2010, Rosasco et al., 2013], learning of piecewise-smooth functions [Lauer et al., 2012], multi-task gradient learning [Ying et al., 2012], structure optimization in parameter-varying ARX processes [Duijkers et al., 2014], density estimation with infinite-dimensional exponential families [Sriperumbudur et al., 2017], Bayesian inference (adaptive samplers) [Strathmann et al., 2015].

In more detail

1. Hermite learning with gradient data:

$$\min_{f \in \mathcal{H}_k} C(f) := \frac{1}{N} \sum_{n \in [N]} \left([f(\mathbf{x}_n) - y_n]^2 + \left\| f'(\mathbf{x}_n) - \mathbf{y}'_n \right\|_2^2 \right) + \lambda \left\| f \right\|_{\mathcal{H}_k}^2 .$$

In more detail

1. Hermite learning with gradient data:

$$\min_{f\in\mathcal{H}_k} C(f) := \frac{1}{N}\sum_{n\in[N]}\left([f(\mathbf{x}_n) - y_n]^2 + \left\|f'(\mathbf{x}_n) - \mathbf{y}'_n\right\|_2^2\right) + \lambda\left\|f\right\|_{\mathcal{H}_k}^2.$$

2. Nonlinear variable selection:

$$\min_{f\in\mathcal{H}_k} C(f) := \frac{1}{N}\sum_{n\in[N]}[f(\mathbf{x}_n) - y_n]^2 + \sum_{j\in[d]}\left\|\partial_j f\right\|,$$

$$\|g\| = \sqrt{\frac{1}{N}\sum_{n\in[N]}|g(x_n)|^2}.$$

In more detail

1. Hermite learning with gradient data:

$$\min_{f\in\mathcal{H}_k} C(f) := \frac{1}{N}\sum_{n\in[N]}\left([f(\mathbf{x}_n) - y_n]^2 + \left\|f'(\mathbf{x}_n) - \mathbf{y}_n'\right\|_2^2\right) + \lambda\left\|f\right\|_{\mathcal{H}_k}^2.$$

2. Nonlinear variable selection:

$$\min_{f\in\mathcal{H}_k} C(f) := \frac{1}{N}\sum_{n\in[N]}[f(\mathbf{x}_n) - y_n]^2 + \sum_{j\in[d]}\left\|\partial_j f\right\|,$$

$$\|g\| = \sqrt{\frac{1}{N}\sum_{n\in[N]}|g(x_n)|^2}.$$

3. Exponential family:

$$p_{\boldsymbol{\theta}}(\mathbf{x}) \propto e^{\left\langle \boldsymbol{\theta}, \overbrace{\mathbf{T}(\mathbf{x})}^{\text{sufficient statistics}} \right\rangle}$$

In more detail

1. Hermite learning with gradient data:

$$\min_{f \in \mathcal{H}_k} C(f) := \frac{1}{N} \sum_{n \in [N]} \left([f(\mathbf{x}_n) - y_n]^2 + \left\| f'(\mathbf{x}_n) - \mathbf{y}'_n \right\|_2^2 \right) + \lambda \left\| f \right\|_{\mathcal{H}_k}^2 .$$

2. Nonlinear variable selection:

$$\min_{f \in \mathcal{H}_k} C(f) := \frac{1}{N} \sum_{n \in [N]} [f(\mathbf{x}_n) - y_n]^2 + \sum_{j \in [d]} \left\| \partial_j f \right\| ,$$

$$\|g\| = \sqrt{\frac{1}{N} \sum_{n \in [N]} |g(x_n)|^2}.$$

3. Infinite-dimensional exponential family (score matching):

$$p_{\boldsymbol{\theta}}(\mathbf{x}) \propto e^{\left\langle \boldsymbol{\theta}, \overbrace{\mathbf{T}(\mathbf{x})}^{\text{sufficient statistics}} \right\rangle} \Rightarrow p_f(\mathbf{x}) \propto e^{\langle f, k(\cdot, \mathbf{x}) \rangle_{\mathcal{H}_k}}$$

In more detail

1. Hermite learning with gradient data:

$$\min_{f\in\mathcal{H}_k} C(f) := \frac{1}{N}\sum_{n\in[N]}\left([f(\mathbf{x}_n)-y_n]^2+\left\|f'(\mathbf{x}_n)-\mathbf{y}'_n\right\|_2^2\right)+\lambda\left\|f\right\|_{\mathcal{H}_k}^2.$$

2. Nonlinear variable selection:

$$\min_{f\in\mathcal{H}_k} C(f) := \frac{1}{N}\sum_{n\in[N]}[f(\mathbf{x}_n)-y_n]^2+\sum_{j\in[d]}\left\|\partial_j f\right\|,$$

$$\|g\| = \sqrt{\frac{1}{N}\sum_{n\in[N]}|g(x_n)|^2}.$$

3. Infinite-dimensional exponential family (score matching):

$$p_{\boldsymbol{\theta}}(\mathbf{x}) \propto e^{\left\langle \boldsymbol{\theta}, \overbrace{\mathbf{T}(\mathbf{x})}^{\text{sufficient statistics}} \right\rangle} \Rightarrow p_f(\mathbf{x}) \propto e^{\langle f, k(\cdot,\mathbf{x})\rangle_{\mathcal{H}_k}} = e^{f(\mathbf{x})} \quad (f\in\mathcal{H}_k).$$

Solution

- Representer theorem [Zhou, 2008]:

$$f(\cdot) = \sum_{\substack{n \in [N] \\ \mathbf{p} \in D_n}} \underbrace{a_{n,\mathbf{p}}}_{\in \mathbb{R}} \partial^{\mathbf{p},\mathbf{0}} k(\cdot, \mathbf{x}_n) \Rightarrow$$

$$\min_{\mathbf{a}} C \left(\left\{ \sum_{\substack{m \in [N] \\ \mathbf{q} \in D_m}} a_{m,\mathbf{q}} \boxed{\partial^{\mathbf{p},\mathbf{q}} k(\mathbf{x}_n, \mathbf{x}_m)} \right\}_{\substack{n \in [N] \\ \mathbf{p} \in D_n}}, \sum_{\substack{n,m \in [N] \\ \mathbf{p} \in D_n \\ \mathbf{q} \in D_m}} a_{n,\mathbf{p}} a_{m,\mathbf{q}} \boxed{\partial^{\mathbf{p},\mathbf{q}} k(\mathbf{x}_n, \mathbf{x}_m)} \right)$$

Solution

- Representer theorem [Zhou, 2008]:

$$f(\cdot) = \sum_{\substack{n\in[N]\\ \mathbf{p}\in D_n}} \underbrace{a_{n,\mathbf{p}}}_{\in\mathbb{R}} \partial^{\mathbf{p},\mathbf{0}} k(\cdot, \mathbf{x}_n) \Rightarrow$$

$$\min_{\mathbf{a}} C\left(\left\{\sum_{\substack{m\in[N]\\ \mathbf{q}\in D_m}} a_{m,\mathbf{q}} \boxed{\partial^{\mathbf{p},\mathbf{q}} k(\mathbf{x}_n, \mathbf{x}_m)}\right\}_{\substack{n\in[N]\\ \mathbf{p}\in D_n}}, \sum_{\substack{n,m\in[N]\\ \mathbf{p}\in D_n\\ \mathbf{q}\in D_m}} a_{n,\mathbf{p}} a_{m,\mathbf{q}} \boxed{\partial^{\mathbf{p},\mathbf{q}} k(\mathbf{x}_n, \mathbf{x}_m)}\right)$$

- RFF [Rahimi and Recht, 2007] with $\boxed{k(\mathbf{x}, \mathbf{x}')} \approx \langle \phi(\mathbf{x}), \phi(\mathbf{x}')\rangle_{\mathbb{R}^M}$

$$f(\mathbf{x}) = \langle f, k(\cdot, \mathbf{x})\rangle_{\mathcal{H}_k} \quad \to \quad \hat{f}_{\mathbf{w}}(\mathbf{x}) = \langle \mathbf{w}, \phi(\mathbf{x})\rangle_{\mathbb{R}^M},$$

Estimate $\mathbf{w}$ by leveraging fast linear primal solvers.

Spectral measure & RFF features

For continuous, bounded, shift-invariant k: Bochner theorem $\Rightarrow$

$$\boxed{k}(\mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^d} e^{i\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})} \mathrm{d}\boxed{\Lambda}(\boldsymbol{\omega})$$

Spectral measure & RFF features

For continuous, bounded, shift-invariant k: Bochner theorem $\Rightarrow$

$$\boxed{k}(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^d} \underbrace{e^{i\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})}}_{\cos\left(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})\right)+i\sin\left(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})\right)} \mathrm{d}\boxed{\Lambda}(\boldsymbol{\omega})$$

Spectral measure & RFF features

For continuous, bounded, shift-invariant k: Bochner theorem $\Rightarrow$

$$\boxed{k}(\mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^d} \cos\left(\boldsymbol{\omega}^T(\mathbf{x} - \mathbf{y})\right) \mathrm{d}\boxed{\Lambda}(\boldsymbol{\omega}).$$

Spectral measure & RFF features

For continuous, bounded, shift-invariant k: Bochner theorem $\Rightarrow$

$$\boxed{k}(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^d} \underbrace{\cos\left(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})\right)}_{\cos(\boldsymbol{\omega}^T\mathbf{x})\cos(\boldsymbol{\omega}^T\mathbf{y})+\sin(\boldsymbol{\omega}^T\mathbf{x})\sin(\boldsymbol{\omega}^T\mathbf{y})} \mathrm{d}\boxed{\Lambda}(\boldsymbol{\omega}).$$

Spectral measure & RFF features

For continuous, bounded, shift-invariant k: Bochner theorem $\Rightarrow$

$$\boxed{k}(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^d} \underbrace{\cos\left(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})\right)}_{\cos(\boldsymbol{\omega}^T\mathbf{x})\cos(\boldsymbol{\omega}^T\mathbf{y})+\sin(\boldsymbol{\omega}^T\mathbf{x})\sin(\boldsymbol{\omega}^T\mathbf{y})} \mathrm{d}\boxed{\Lambda}(\boldsymbol{\omega}).$$

Trick: $(\boldsymbol{\omega}_m)_{m=1}^M \overset{\text{i.i.d.}}{\sim} \Lambda$,

$$\boxed{\hat{k}}(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^d} \cos\left(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})\right) \mathrm{d}\boxed{\Lambda_M}(\boldsymbol{\omega})$$

Spectral measure & RFF features

For continuous, bounded, shift-invariant k: Bochner theorem $\Rightarrow$

$$\boxed{k}(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^d} \underbrace{\cos\left(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})\right)}_{\cos(\boldsymbol{\omega}^T\mathbf{x})\cos(\boldsymbol{\omega}^T\mathbf{y})+\sin(\boldsymbol{\omega}^T\mathbf{x})\sin(\boldsymbol{\omega}^T\mathbf{y})} \mathrm{d}\boxed{\Lambda}(\boldsymbol{\omega}).$$

Trick: $(\boldsymbol{\omega}_m)_{m=1}^M \overset{\text{i.i.d.}}{\sim} \Lambda$,

$$\begin{aligned}\boxed{\hat{k}}(\mathbf{x},\mathbf{y}) &= \int_{\mathbb{R}^d} \cos\left(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})\right) \mathrm{d}\boxed{\Lambda_M}(\boldsymbol{\omega}) \\ &= \langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle, \\ \phi(\mathbf{x}) &= \frac{1}{\sqrt{M}}\left[\left(\cos(\boldsymbol{\omega}_m^T\mathbf{x})\right)_{m=1}^M, \left(\sin(\boldsymbol{\omega}_m^T\mathbf{x})\right)_{m=1}^M\right] \in \mathbb{R}^{2M}.\end{aligned}$$

Spectral measure & RFF features

For continuous, bounded, shift-invariant k: Bochner theorem $\Rightarrow$

$$\boxed{k}(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^d} \underbrace{\cos\left(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})\right)}_{\cos(\boldsymbol{\omega}^T\mathbf{x})\cos(\boldsymbol{\omega}^T\mathbf{y})+\sin(\boldsymbol{\omega}^T\mathbf{x})\sin(\boldsymbol{\omega}^T\mathbf{y})} \,\mathrm{d}\boxed{\Lambda}(\boldsymbol{\omega}).$$

Trick: $(\boldsymbol{\omega}_m)_{m=1}^M \overset{\text{i.i.d.}}{\sim} \Lambda$,

$$\begin{aligned}\boxed{\hat{k}}(\mathbf{x},\mathbf{y}) &= \int_{\mathbb{R}^d} \cos\left(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})\right) \mathrm{d}\boxed{\Lambda_M}(\boldsymbol{\omega}) \\ &= \langle \phi(\mathbf{x}), \phi(\mathbf{y})\rangle, \\ \phi(\mathbf{x}) &= \frac{1}{\sqrt{M}}\left[\left(\cos(\boldsymbol{\omega}_m^T\mathbf{x})\right)_{m=1}^M, \left(\sin(\boldsymbol{\omega}_m^T\mathbf{x})\right)_{m=1}^M\right] \in \mathbb{R}^{2M}.\end{aligned}$$

$\widehat{\partial^{\mathbf{p},\mathbf{q}} k}$ similarly.

RFF applications

10-year test-of-time award (NIPS-2017).

RFF applications

10-year test-of-time award (NIPS-2017).

Examples: differential privacy preserving [Chaudhuri et al., 2011], fast function-to-function regression [Oliva et al., 2015], learning message operators in expectation propagation [Jitkrittum et al., 2015], causal discovery [Lopez-Paz et al., 2015, Strobl et al., 2019], independence testing [Zhang et al., 2017], prediction and filtering in dynamical systems [Downey et al., 2017], bandit optimization [Li et al., 2018], estimation of Gaussian mixture models [Keriven et al., 2018].

RFF guarantee: k

- Kernel values
  [Rahimi and Recht, 2007, Sutherland and Schneider, 2015]

$$\left\|k - \widehat{k}\right\|_{L^{\infty}(S_M)} = \mathcal{O}_p\left(|S_M|\sqrt{\frac{\log M}{M}}\right)$$

RFF guarantee: k

- Kernel values
  [Rahimi and Recht, 2007, Sutherland and Schneider, 2015],
  [Csörgö and Totik, 1983]

$$\left\|k - \widehat{k}\right\|_{L^\infty(S_M)} = \mathcal{O}_p\left(|S_M|\sqrt{\frac{\log M}{M}}\right), |S_M| = e^{o(M)} \text{ is expected} \Rightarrow$$

RFF guarantee: k

- Kernel values
  [Rahimi and Recht, 2007, Sutherland and Schneider, 2015], [Csörgö and Totik, 1983], [Sriperumbudur and Szabó, 2015]:

$$\left\|k - \widehat{k}\right\|_{L^\infty(S_M)} = \mathcal{O}_p\left(|S_M|\sqrt{\frac{\log M}{M}}\right), |S_M| = e^{o(M)} \text{ is expected} \Rightarrow$$

$$\left\|k - \widehat{k}\right\|_{L^\infty(S_M)} = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\log|S_M|}{M}}\right).$$

RFF guarantee: k

- Downstream tasks:
  1. Kernel ridge regression [Rudi and Rosasco, 2017], [Li et al., 2019]:
     - $\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$ generalization with $M = o(N) = \mathcal{O}\left(\sqrt{N}\log N\right)$ or less RFFs.

RFF guarantee: k

- Downstream tasks:
  1. Kernel ridge regression [Rudi and Rosasco, 2017], [Li et al., 2019]:
     - $\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$ generalization with $M = o(N) = \mathcal{O}\left(\sqrt{N}\log N\right)$ or less RFFs.
  2. Kernel PCA [Sriperumbudur and Sterge, 2018, Ullah et al., 2018], classification with 0-1 loss [Gilbert et al., 2018]: $M = o(N)$ RFFs, spectrum decay.

Challenge

- Kernel values ($\mathbf{p} = \mathbf{q} = \mathbf{0}$):

$$\left\| \hat{k} - k \right\|_{L^\infty(S)} = \sup_{f \in \mathcal{F}} |(\Lambda_M - \Lambda)(f)|, \quad \mathcal{F} = \{ f_{\mathbf{z}} : \mathbf{z} \in S_\Delta := S - S \},$$

$$f_{\mathbf{z}}(\boldsymbol{\omega}) = \cos\left(\boldsymbol{\omega}^\top \mathbf{z}\right).$$

$\mathcal{F}$ is uniformly bounded: $\sup_{\mathbf{z} \in S_\Delta} \|f_{\mathbf{z}}\|_{L^\infty(\mathbb{R}^d)} < \infty$.

Challenge

- Kernel values ($\mathbf{p} = \mathbf{q} = \mathbf{0}$):

$$\left\|\hat{k} - k\right\|_{L^\infty(S)} = \sup_{f \in \mathcal{F}} |(\Lambda_M - \Lambda)(f)|, \quad \mathcal{F} = \{f_{\mathbf{z}} : \mathbf{z} \in S_\Delta := S - S\},$$

$$f_{\mathbf{z}}(\boldsymbol{\omega}) = \cos\left(\boldsymbol{\omega}^\top \mathbf{z}\right).$$

$\mathcal{F}$ is uniformly bounded: $\sup_{\mathbf{z} \in S_\Delta} \|f_{\mathbf{z}}\|_{L^\infty(\mathbb{R}^d)} < \infty$.

- Kernel derivatives ($[\mathbf{p}, \mathbf{q}] \neq \mathbf{0}$): $\left\|\widehat{\partial^{\mathbf{p},\mathbf{q}} k} - \partial^{\mathbf{p},\mathbf{q}} k\right\|_{L^\infty(S)}$ with

$$f_{\mathbf{z}}(\boldsymbol{\omega}) = \boxed{\boldsymbol{\omega}^{\mathbf{p}}(-\boldsymbol{\omega})^{\mathbf{q}}} \cos^{(|\mathbf{p}+\mathbf{q}|)}\left(\boldsymbol{\omega}^\top \mathbf{z}\right), \qquad \boldsymbol{\omega}^{\mathbf{p}} = \prod_{i=1}^{d} \omega_i^{p_i}.$$

Challenge

- Kernel values ($\mathbf{p} = \mathbf{q} = \mathbf{0}$):

$$\left\|\hat{k} - k\right\|_{L^\infty(S)} = \sup_{f \in \mathcal{F}} |(\Lambda_M - \Lambda)(f)|, \quad \mathcal{F} = \{f_{\mathbf{z}} : \mathbf{z} \in S_\Delta := S - S\},$$
$$f_{\mathbf{z}}(\boldsymbol{\omega}) = \cos\left(\boldsymbol{\omega}^\top \mathbf{z}\right).$$

$\mathcal{F}$ is uniformly bounded: $\sup_{\mathbf{z} \in S_\Delta} \|f_{\mathbf{z}}\|_{L^\infty(\mathbb{R}^d)} < \infty$.

- Kernel derivatives ($[\mathbf{p}, \mathbf{q}] \neq \mathbf{0}$): $\left\|\widehat{\partial^{\mathbf{p},\mathbf{q}} k} - \partial^{\mathbf{p},\mathbf{q}} k\right\|_{L^\infty(S)}$ with

$$f_{\mathbf{z}}(\boldsymbol{\omega}) = \boxed{\boldsymbol{\omega}^{\mathbf{p}}(-\boldsymbol{\omega})^{\mathbf{q}}} \cos^{(|\mathbf{p}+\mathbf{q}|)}\left(\boldsymbol{\omega}^\top \mathbf{z}\right), \qquad \boldsymbol{\omega}^{\mathbf{p}} = \prod_{i=1}^{d} \omega_i^{p_i}.$$

$\mathcal{F}$ is of **polynomial growth**.

RFF guarantee: $\partial^{\mathbf{p},\mathbf{q}} k$

Kernel derivatives [Szabó and Sriperumbudur, 2019]:

- Same fast rate as for kernel values (unbounded emp. processes).

RFF guarantee: $\partial^{\mathbf{p},\mathbf{q}} k$

Kernel derivatives [Szabó and Sriperumbudur, 2019]:

- Same fast rate as for kernel values (unbounded emp. processes).
- Bernstein condition on Λ: $d = 1$, $f_\Lambda(\omega) \propto e^{-\omega^{2\ell}} \Rightarrow p + q \leq 2\ell : \checkmark$

RFF guarantee: $\partial^{\mathbf{p},\mathbf{q}} k$

Now: α-exponential Orlicz spectrum (Bernstein $\Rightarrow$ sub-exponential)

f_Λ spectrum with at least $e^{-\|\boldsymbol{\omega}\|_2^\alpha}$ tail decay, $\alpha > 0$.

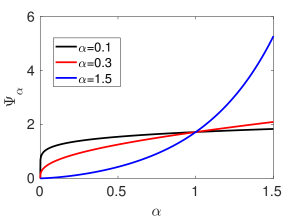
- Examples: sub-Gaussian ($\alpha = 2$), sub-exponential ($\alpha = 1$).

RFF guarantee: $\partial^{\mathbf{p},\mathbf{q}} k$

Now: α-exponential Orlicz spectrum (Bernstein $\Rightarrow$ sub-exponential)

f_Λ spectrum with at least $e^{-\|\boldsymbol{\omega}\|_2^\alpha}$ tail decay, $\alpha > 0$.

- Examples: sub-Gaussian ($\alpha = 2$), sub-exponential ($\alpha = 1$).
- $L_{\Psi_\alpha} := \left\{ \Lambda : \|\Lambda\|_{\Psi_\alpha} := \inf\left\{ c > 0 : \mathbb{E}_{\boldsymbol{\omega}\sim\Lambda} \Psi_\alpha\left(\frac{\|\boldsymbol{\omega}\|_2}{c}\right) \le 1 \right\} < +\infty \right\}$.
- $\Psi_\alpha : x \in \mathbb{R}^{\ge 0} \mapsto e^{x^\alpha} - 1 \in \mathbb{R}^{\ge 0}$.



Main result

Blanket assumptions:

- $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ continuous, bounded, shift-invariant kernel with α-exponential Orlicz spectrum ($\alpha > 0$),
- $\mathbf{p}, \mathbf{q} \in \mathbb{N}^d$.

Main result

Blanket assumptions:

- $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ continuous, bounded, shift-invariant kernel with α-exponential Orlicz spectrum ($\alpha > 0$),
- $\mathbf{p}, \mathbf{q} \in \mathbb{N}^d$.

Finite sample guarantee [Chamakh et al., 2019], $\Rightarrow$

Fast rates

$$\left\| \partial^{\mathbf{p},\mathbf{q}} k - \widehat{\partial^{\mathbf{p},\mathbf{q}} k} \right\|_{L^\infty(S_M)} = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\log |S_M|}{M}}\right), \Rightarrow |S_M| = e^{o(M)} \checkmark$$

α-exponential Orlicz factory

For tensor product kernels:

If

- $k_i \leftrightarrow \Lambda_i \in L_{\Psi_{\alpha_i}}$

For tensor product kernels:

If

- $k_i \leftrightarrow \Lambda_i \in L_{\Psi_{\alpha_i}}$
- $k(\mathbf{x}, \mathbf{y}) = \prod_{i \in [d]} k_i(x_i, y_i)$, i.e. $\Lambda = \otimes_{i \in [d]} \Lambda_i$,

α-exponential Orlicz factory

For tensor product kernels:

If

- $k_i \leftrightarrow \Lambda_i \in L_{\Psi_{\alpha_i}}$ and
- $k(\mathbf{x}, \mathbf{y}) = \prod_{i \in [d]} k_i(x_i, y_i)$, i.e. $\Lambda = \otimes_{i \in [d]} \Lambda_i$,

then $\Lambda \in L_{\Psi_\alpha}$ with $\alpha = \min_{i \in [d]} \alpha_i$.

Kernel examples with α-exp. Orlicz spectrum: $d = 1$

Spectrum	$f_\Lambda(\omega)$	α
Gaussian	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\omega^2}{2\sigma^2}}$	2

Kernel examples with α-exp. Orlicz spectrum: $d = 1$

Spectrum	$f_\Lambda(\omega)$	α
Gaussian	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\omega^2}{2\sigma^2}}$	2
Laplace	$\frac{\sigma}{2} e^{-\sigma\lvert\omega\rvert}$	1

Kernel examples with α-exp. Orlicz spectrum: $d = 1$

Spectrum	$f_\Lambda(\omega)$	α
Gaussian	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\omega^2}{2\sigma^2}}$	2
Laplace	$\frac{\sigma}{2} e^{-\sigma\lvert\omega\rvert}$	1
generalized Gaussian	$\frac{\alpha}{2\beta\Gamma\left(\frac{1}{\alpha}\right)} e^{-\left\lvert\frac{\omega}{\beta}\right\rvert^{\alpha}}$	α

$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x.$

Kernel examples with α-exp. Orlicz spectrum: $d = 1$

Spectrum	$f_\Lambda(\omega)$	α
Gaussian	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\omega^2}{2\sigma^2}}$	2
Laplace	$\frac{\sigma}{2} e^{-\sigma\lvert\omega\rvert}$	1
generalized Gaussian	$\frac{\alpha}{2\beta\Gamma\left(\frac{1}{\alpha}\right)} e^{-\frac{\lvert\omega\rvert}{\beta}^{\alpha}}$	α
variance Gamma	$\frac{\sigma^{2b}\lvert\omega\rvert^{b-\frac{1}{2}} K_{b-\frac{1}{2}}(\sigma\lvert\omega\rvert)}{\sqrt{\pi}\Gamma(b)(2\sigma)^{b-\frac{1}{2}}}$	1

$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x$. K_b: modified Bessel function of 2nd kind and order b.

Kernel examples with α-exp. Orlicz spectrum: $d = 1$

Spectrum	$f_\Lambda(\omega)$	α
Gaussian	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\omega^2}{2\sigma^2}}$	2
Laplace	$\frac{\sigma}{2} e^{-\sigma\|\omega\|}$	1
generalized Gaussian	$\frac{\alpha}{2\beta\Gamma\left(\frac{1}{\alpha}\right)} e^{-\frac{\|\omega\|}{\beta}^{\alpha}}$	α
variance Gamma	$\frac{\sigma^{2b}\|\omega\|^{b-\frac{1}{2}} K_{b-\frac{1}{2}}(\sigma\|\omega\|)}{\sqrt{\pi}\Gamma(b)(2\sigma)^{b-\frac{1}{2}}}$	1
hyperbolic secant	$\frac{1}{2}\operatorname{sech}\left(\frac{\pi}{2}\omega\right)$	1

$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x$. K_b: modified Bessel function of 2nd kind and order b. $\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$.

Kernel examples with α-exp. Orlicz spectrum: $d = 1$

Spectrum	$f_\Lambda(\omega)$	α
Gaussian	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\omega^2}{2\sigma^2}}$	2
Laplace	$\frac{\sigma}{2} e^{-\sigma\lvert\omega\rvert}$	1
generalized Gaussian	$\frac{\alpha}{2\beta\Gamma\left(\frac{1}{\alpha}\right)} e^{-\frac{\lvert\omega\rvert}{\beta}^{\alpha}}$	α
variance Gamma	$\frac{\sigma^{2b}\lvert\omega\rvert^{b-\frac{1}{2}} K_{b-\frac{1}{2}}(\sigma\lvert\omega\rvert)}{\sqrt{\pi}\Gamma(b)(2\sigma)^{b-\frac{1}{2}}}$	1
hyperbolic secant	$\frac{1}{2}\operatorname{sech}\left(\frac{\pi}{2}\omega\right)$	1
logistic	$\frac{e^{-\frac{\omega}{s}}}{s\left[1+e^{-\frac{\omega}{s}}\right]^2} = \frac{1}{4s}\operatorname{sech}^2\left(\frac{\omega}{2s}\right)$	1

$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x$. K_b: modified Bessel function of 2nd kind and order b. $\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x+e^{-x}}$. 23 examples in TR.

Kernel examples $\leftarrow\!\!\dashv$ spectrum ($b > \frac{1}{2}$, $s > 0$)

Kernel	$k(x,y)$	Spectrum
Gaussian	$e^{-\frac{\sigma^2(x-y)^2}{2}}$	Gaussian
Cauchy / inverse quadric	$\frac{\sigma^2}{\sigma^2+(x-y)^2}$	Laplace
inverse multiquadric	$\left[\frac{\sigma^2}{\sigma^2+(x-y)^2}\right]^b$	variance Gamma
–	$\operatorname{sech}(x-y)$	hyperbolic secant
–	$\frac{\pi s(x-y)}{\sinh(\pi s(x-y))}$	logistic

Summary

- Focus: RFF-based acceleration for derivatives.
- Result: α-exponential Orlicz spectrum $\Rightarrow$ fast rates ($\forall \mathbf{p}, \mathbf{q}$ order),
- Preprint on HAL: Orlicz Random Fourier Features.
- Future: downstream tasks.

Summary

- Focus: RFF-based acceleration for derivatives.
- Result: α-exponential Orlicz spectrum $\Rightarrow$ fast rates ($\forall \mathbf{p}, \mathbf{q}$ order),
- Preprint on HAL: Orlicz Random Fourier Features.
- Future: downstream tasks.

Stress Test
RISK Management and Financial Steering

Acknowledgments: This research benefited from the support of the Chair Stress Test, RISK Management and Financial Steering, led by the French Ecole polytechnique and its Foundation and sponsored by BNP Paribas.

Summary

- Focus: RFF-based acceleration for derivatives.
- Result: α-exponential Orlicz spectrum $\Rightarrow$ fast rates ($\forall \mathbf{p}, \mathbf{q}$ order),
- Preprint on HAL: Orlicz Random Fourier Features.
- Future: downstream tasks.

Stress Test
RISK Management and Financial Steering

Acknowledgments: This research benefited from the support of the Chair Stress Test, RISK Management and Financial Steering, led by the French Ecole polytechnique and its Foundation and sponsored by BNP Paribas.

Contents

- Kernel examples with α-exponential Orlicz spectrum.
- Proof idea.

Kernel examples with α-exp. Orlicz spectrum: $d = 1$

Spectrum	$f_\Lambda(\omega)$	α
Gaussian	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\omega^2}{2\sigma^2}}$	2
Laplace	$\frac{\sigma}{2} e^{-\sigma\|\omega\|}$	1
generalized Gaussian	$\frac{\alpha}{2\beta\Gamma\left(\frac{1}{\alpha}\right)} e^{-\frac{\|\omega\|}{\beta}^{\alpha}}$	α
variance Gamma	$\frac{\sigma^{2b}\|\omega\|^{b-\frac{1}{2}} K_{b-\frac{1}{2}}(\sigma\|\omega\|)}{\sqrt{\pi}\Gamma(b)(2\sigma)^{b-\frac{1}{2}}}$	1
Weibull (S)	$\frac{s}{2\lambda}\left(\frac{\|\omega\|}{\lambda}\right)^{s-1} e^{-\left(\frac{\|\omega\|}{\lambda}\right)^s}$	s
exponentiated exponential (S)	$\frac{\alpha}{2\lambda}\left(1 - e^{-\frac{\|\omega\|}{\lambda}}\right)^{\alpha-1} e^{-\frac{\|\omega\|}{\lambda}}$	1

$I_a(z) = \sum_{n\in\mathbb{N}} \frac{1}{n!\Gamma(n+a+1)}\left(\frac{z}{2}\right)^{2n+a}$, $K_a(z) = \frac{\pi}{2}\frac{I_{-a}(z)-I_a(z)}{\sin(a\pi)}$ for $z \in \mathbb{R}$ and non-integer a; when a is an integer the limit is taken.

Kernel examples with α-exponential Orlicz spectrum - 2

Spectrum	$f_\Lambda(\omega)$	α
exponentiated Weibull (S)	$\frac{\alpha s}{2\lambda}\left(\frac{\lvert\omega\rvert}{\lambda}\right)^{s-1}\left[1-e^{-\left(\frac{\lvert\omega\rvert}{\lambda}\right)^s}\right]^{\alpha-1}\times$ $\times e^{-\left(\frac{\lvert\omega\rvert}{\lambda}\right)^s}$	s
Nakagami (S)	$\frac{m^m}{\Gamma(m)\Omega^m}\lvert\omega\rvert^{2m-1}e^{-\frac{m\omega^2}{\Omega}}$	2
chi-squared (S)	$\frac{1}{2^{\frac{s}{2}+1}\Gamma\left(\frac{s}{2}\right)}\lvert\omega\rvert^{\frac{s}{2}-1}e^{-\frac{\lvert\omega\rvert}{2}}$	1
Erlang (S)	$\frac{\lambda^s\lvert\omega\rvert^{s-1}e^{-\lambda\lvert\omega\rvert}}{2(s-1)!}$	1
Gamma (S)	$\frac{1}{2\Gamma(s)\theta^s}\lvert\omega\rvert^{s-1}e^{-\frac{\lvert\omega\rvert}{\theta}}$	1
generalized Gamma (S)	$\frac{p/a^D}{2\Gamma\left(\frac{D}{p}\right)}\lvert\omega\rvert^{D-1}e^{-\left(\frac{\lvert\omega\rvert}{a}\right)^p}$	p

Kernel examples with α-exponential Orlicz spectrum - 3

Spectrum	$f_\Lambda(\omega)$	α
Rayleigh (S)	$\frac{\lvert\omega\rvert}{2\sigma^2} e^{-\frac{\omega^2}{2\sigma^2}}$	2
Maxwell-Boltzmann (S)	$\frac{1}{\sqrt{2\pi}} \frac{\omega^2 e^{-\frac{\omega^2}{2a^2}}}{a^3}$	2
chi (S)	$\frac{1}{2^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)} \lvert\omega\rvert^{s-1} e^{-\frac{\omega^2}{2}}$	2
exponential-logarithmic (S)	$-\frac{1}{2\log(p)} \frac{\beta(1-p)e^{-\beta\lvert\omega\rvert}}{1-(1-p)e^{-\beta\lvert\omega\rvert}}$	1
Weibull-logarithmic (S)	$-\frac{1}{2\log(p)} \frac{\alpha\beta(1-p)\lvert\omega\rvert^{\alpha-1} e^{-\beta\lvert\omega\rvert^\alpha}}{1-(1-p)e^{-\beta\lvert\omega\rvert^\alpha}}$	α
Gamma/Gompertz (S)	$\frac{bse^{b\lvert\omega\rvert}\beta^s}{2\left(\beta-1+e^{b\lvert\omega\rvert}\right)^{s+1}}$	bs

Kernel examples with α-exponential Orlicz spectrum - 4

Spectrum	$f_\Lambda(\omega)$	α
hyperbolic secant	$\frac{1}{2}\operatorname{sech}\left(\frac{\pi}{2}\omega\right)$	1
logistic	$\frac{e^{-\frac{\omega}{s}}}{s\left[1+e^{-\frac{\omega}{s}}\right]^2} = \frac{1}{4s}\operatorname{sech}^2\left(\frac{\omega}{2s}\right)$	1
normal-inverse Gaussian	$\frac{\alpha\delta K_1\left(\alpha\sqrt{\delta^2+\omega^2}\right)}{\pi\sqrt{\delta^2+\omega^2}} e^{\delta\alpha}$	1
hyperbolic	$\frac{1}{2\delta K_1(\delta\alpha)} e^{-\alpha\sqrt{\delta^2+\omega^2}}$	1
generalized hyperbolic	$\frac{(\alpha/\delta)^\lambda}{\sqrt{2\pi}K_\lambda(\delta\gamma)} \frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+\omega^2}\right)}{\left(\frac{\sqrt{\delta^2+\omega^2}}{\alpha}\right)^{\frac{1}{2}-\lambda}}$	1

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x+e^{-x}}.$$

Proof idea

Decomposition into 3 terms:

1. Unbounded part: Talagrand & Hoffman-Jorgensen inequalities.
2. Bounded part: Klein-Rio inequality & Dudley entropy integral bound.
3. Truncation: bound on the incomplete Gamma function.

Chamakh, L., Gobet, E., and Szabó, Z. (2019).
Orlicz random Fourier features.
Technical report.
(https://hal.archives-ouvertes.fr/hal-02418576).

Chaudhuri, K., Monteleoni, C., and Sarwate, A. D. (2011).
Differentially private empirical risk minimization.
Journal of Machine Learning Research, 12:1069–1109.

Csörgö, S. and Totik, V. (1983).
On how long interval is the empirical characteristic function uniformly consistent?
Acta Scientiarum Mathematicarum, 45:141–149.

Downey, C., Hefny, A., Boots, B., Li, B., and Gordon, G. (2017).
Predictive state recurrent neural networks.
In *Advances in Neural Information Processing Systems (NIPS)*, pages 6053–6064.

Duijkers, R., Tóth, R., Piga, D., and Laurain, V. (2014).
Shrinking complexity of scheduling dependencies in LS-SVM based LPV system identification.
In *IEEE Conference on Decision and Control*, pages 2561–2566.

Gilbert, A., Tewari, A., and Sung, Y. (2018).
But how does it work in theory? Linear SVM with random features.
In *Advances in Neural Information Processing Systems (NeurIPS)*, pages 3379–3388.

Jitkrittum, W., Gretton, A., Heess, N., Eslami, A., Lakshminarayanan, B., Sejdinovic, D., and Szabó, Z. (2015).
Kernel-based just-in-time learning for passing expectation propagation messages.
In *Conference on Uncertainty in Artificial Intelligence (UAI)*, pages 405–414.

Keriven, N., Bourrier, A., Gribonval, R., and Pérez, P. (2018).

Sketching for large-scale learning of mixture models.
Information and Inference: A Journal of the IMA, 7:447–508.

Lauer, F., Le, V. L., and Bloch, G. (2012).
Learning smooth models of nonsmooth functions via convex optimization.
In *International Workshop on Machine Learning for Signal Processing (IEEE-MLSP)*.

Li, L., Jamieson, K., DeSalvo, G., Rostamizadeh, A., and Talwalkar, A. (2018).
Hyperband: A novel bandit-based approach to hyperparameter optimization.
Journal of Machine Learning Research, 18(185):1–52.

Li, Z., Ton, J.-F., Oglic, D., and Sejdinovic, D. (2019).
A unified analysis of random Fourier features.
In *International Conference on Machine Learning (ICML; PMLR)*.

Lopez-Paz, D., Muandet, K., Schölkopf, B., and Tolstikhin, I. (2015).
Towards a learning theory of cause-effect inference.
International Conference on Machine Learning (ICML), 37:1452–1461.

Oliva, J., Neiswanger, W., Póczos, B., Xing, E., and Schneider, J. (2015).
Fast function to function regression.
In *International Conference on Artificial Intelligence and Statistics (AISTATS; PMLR)*, pages 717–725.

Rahimi, A. and Recht, B. (2007).
Random features for large-scale kernel machines.
In *Advances in Neural Information Processing Systems (NIPS)*, pages 1177–1184.

Rosasco, L., Santoro, M., Mosci, S., Verri, A., and Villa, S. (2010).
A regularization approach to nonlinear variable selection.

In *International Conference on Artificial Intelligence and Statistics (AISTATS; PMLR)*, volume 9, pages 653–660.

Rosasco, L., Villa, S., Mosci, S., Santoro, M., and Verri, A. (2013).
Nonparametric sparsity and regularization.
Journal of Machine Learning Research, 14:1665–1714.

Rudi, A. and Rosasco, L. (2017).
Generalization properties of learning with random features.
In *Advances in Neural Information Processing Systems (NIPS)*, pages 3218–3228.

Sriperumbudur, B. and Sterge, N. (2018).
Approximate kernel PCA using random features: Computational vs. statistical trade-off.
Technical report, Pennsylvania State University. (https://arxiv.org/abs/1706.06296).

Sriperumbudur, B. K., Fukumizu, K., Gretton, A., Hyvärinen, A., and Kumar, R. (2017).

Density estimation in infinite dimensional exponential families.
Journal of Machine Learning Research, 18(57):1–59.

Sriperumbudur, B. K. and Szabó, Z. (2015).
Optimal rates for random Fourier features.
In *Advances in Neural Information Processing Systems (NIPS)*, pages 1144–1152.

Strathmann, H., Sejdinovic, D., Livingstone, S., Szabó, Z., and Gretton, A. (2015).
Gradient-free Hamiltonian Monte Carlo with efficient kernel exponential families.
In *Advances in Neural Information Processing Systems (NIPS)*, pages 955–963.

Strobl, E. V., Zhang, K., and Visweswaran, S. (2019).
Approximate kernel-based conditional independence tests for fast non-parametric causal discovery.
Journal of Causal Inference, 7(1).
(`https://doi.org/10.1515/jci-2018-0017`).

Sutherland, D. J. and Schneider, J. (2015).
On the error of random Fourier features.
In *Conference on Uncertainty in Artificial Intelligence (UAI)*, pages 862–871.

Szabó, Z. and Sriperumbudur, B. K. (2019).
On kernel derivative approximation with random Fourier features.
In *International Conference on Artificial Intelligence and Statistics (AISTATS; PMLR)*, volume 89, pages 827–836.

Ullah, E., Mianjy, P., Marinov, T. V., and Arora, R. (2018).
Streaming kernel PCA with $\tilde{O}(\sqrt{n})$ random features.
In *Advances in Neural Information Processing Systems (NeurIPS)*, pages 7311–7321.

Ying, Y., Wu, Q., and Campbell, C. (2012).
Learning the coordinate gradients.
Advances in Computational Mathematics, 37:355–378.

Zhang, Q., Filippi, S., Gretton, A., and Sejdinovic, D. (2017).
Large-scale kernel methods for independence testing.
Statistics and Computing, pages 1–18.

Zhou, D.-X. (2008).
Derivative reproducing properties for kernel methods in learning theory.
Journal of Computational and Applied Mathematics, 220:456–463.