Shape Constraints Meet Kernel Machines

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Joint work with: Pierre-Cyril Aubin-Frankowski @ MINES ParisTech



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- 3 convexity: $0 \le f''(x)$,
- *n*-monotonicity: $0 \le f^{(n)}(x)$,
- **3** (n-1)-alternating monotonicity: for $n \ge 2$

$$(-1)^j f^{(j)}$$
: ≥ 0 , \nearrow and convex $\forall j \in \llbracket 0, n-2
rbracket$.

Example: generator of a d-variate Archimedean copula is (d-2)-alternating monotone.

Examples continued

6 Monotonicity w.r.t. partial ordering $(\mathbf{u} \leq \mathbf{v} \Rightarrow f(\mathbf{u}) \leq f(\mathbf{v}))$:

 $\mathbf{u} \preccurlyeq \mathbf{v}$ iff

- $\underline{u_i} \le v_i$ ($\forall i$; product ordering),
- $\sum_{j \in [i]} u_j \leq \sum_{j \in [i]} v_j$ ($\forall i$; unordered weak majorization).

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$$\frac{0 \le \partial^{\mathbf{e}_j} f(\mathbf{x})}{0 \le \partial^{\mathbf{e}_d} f(\mathbf{x}) \le \dots \le \partial^{\mathbf{e}_1} f(\mathbf{x})}, \quad (\forall j \in [d], \forall \mathbf{x}),$$

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- $u_i \le v_i$ ($\forall i$; product ordering),
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- Supermodularity:

$$0 \leq \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \quad (\forall i \neq j \in [d], \forall \mathbf{x}),$$

i.e.
$$f(\mathbf{u} \vee \mathbf{v}) + f(\mathbf{u} \wedge \mathbf{v}) \geq f(\mathbf{u}) + f(\mathbf{v})$$
 for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$.

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- Supply chain models, game theory: supermodularity [Topkis, 1998, Simchi-Levi et al., 2014].

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Towards flexible H-s . . .

Kernel

• Def-1 (feature space): $k: \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ kernel if

$$k(x,y) = \langle \varphi(x), \varphi(y) \rangle_{\mathcal{H}}.$$

• Examples $(\gamma > 0, c \ge 0, p \in \mathbb{Z}^+)$:

$$\begin{aligned} k_p(\mathbf{x}, \mathbf{y}) &= (\langle \mathbf{x}, \mathbf{y} \rangle + c)^p, & k_G(\mathbf{x}, \mathbf{y}) &= e^{-\gamma \|\mathbf{x} - \mathbf{y}\|_2^2}, \\ k_L(\mathbf{x}, \mathbf{y}) &= e^{-\gamma \|\mathbf{x} - \mathbf{y}\|_1}, & k_e(\mathbf{x}, \mathbf{y}) &= e^{\gamma \langle \mathbf{x}, \mathbf{y} \rangle}. \end{aligned}$$

Kernel, RKHS

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Def-2 (reproducing kernel):

$$k(\cdot,x) := [x' \mapsto k(x',x)] \in \mathcal{H}, \qquad f(x) = \langle f, k(\cdot,x) \rangle_{\mathcal{H}}.$$

Constructively, $\mathfrak{H}_k = \overline{\{\sum_{i=1}^n \alpha_i k(\cdot, x_i) : \alpha_i \in \mathbb{R}, x_i \in \mathfrak{X}, n \in \mathbb{N}^*\}}$.

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- Equivalent definitions, $k \stackrel{1:1}{\leftrightarrow} \mathcal{H}_k$.
- Included: Fourier analysis, polynomials, splines, ...
- Examples $(\gamma > 0, c \ge 0, p \in \mathbb{Z}^+)$:

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Kernels on other domains (\mathfrak{X})

- Strings [Watkins, 1999, Lodhi et al., 2002, Leslie et al., 2002, Kuang et al., 2004, Leslie and Kuang, 2004, Saigo et al., 2004, Cuturi and Vert, 2005],
- time series [Rüping, 2001, Cuturi et al., 2007, Cuturi, 2011, Király and Oberhauser, 2019],
- trees [Collins and Duffy, 2001, Kashima and Koyanagi, 2002],
- groups and specifically rankings
 [Cuturi et al., 2005, Jiao and Vert, 2016],
- sets [Haussler, 1999, Gärtner et al., 2002],
- various generative models [Jaakkola and Haussler, 1999, Tsuda et al., 2002, Seeger, 2002, Jebara et al., 2004],
- fuzzy domains [Guevara et al., 2017], or
- graphs [Kondor and Lafferty, 2002, Gärtner et al., 2003, Kashima et al., 2003, Borgwardt and Kriegel, 2005, Shervashidze et al., 2009, Vishwanathan et al., 2010, Kondor and Pan, 2016, Bai et al., 2020, Borgwardt et al., 2020].

- Given: $(\tau_q)_{q \in [Q]} \subset (0,1)$ levels \nearrow , $\{(\mathbf{x}_n, y_n)\}_{n \in [N]}$ samples.
- Estimate jointly the τ_q -quantiles of $\mathbb{P}(Y|X=\mathbf{x})$.

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- Objective:

$$\mathcal{L}(\mathbf{f}, \mathbf{b}) = \underbrace{\frac{1}{N} \sum_{q \in [Q]} \sum_{n \in [N]} l_{\tau_q} \left(y_n - [f_q(\mathbf{x}_n) + b_q] \right)}_{\text{quantile property}} + \underbrace{\lambda_{\mathbf{b}} \|\mathbf{b}\|_2^2 + \lambda_{\mathbf{f}} \sum_{q \in [Q]} \|f_q\|_k^2}_{\text{regularization}},$$

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• Constraint (non-crossing): $K := \text{smallest rectangle containing } \{x_n\}_{n \in [N]}$

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Constraints

function values (f_q) with interaction $(f_{q+1} - f_q)$, bias terms (b_q) with interaction $(b_q - b_{q+1})$.

Task-2: convoy localization, one vehicle (Q = 1)

- Given: noisy time-location samples $\{(t_n, x_n)\}_{n \in [N]} \subset [0, T] \times \mathbb{R}$. Goal: learn the (t, x) relation.
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Task-2: convoy localization, one vehicle (Q = 1)

- Given: noisy time-location samples $\{(t_n, x_n)\}_{n \in [N]} \subset [0, T] \times \mathbb{R}$.
- Goal: learn the (t, x) relation.
- Constraint: lower bound on speed (v_{min}) .
- Objective:

$$\min_{b \in \mathbb{R}, f \in \mathcal{H}_k} \left[\frac{1}{N} \sum_{n \in [N]} |x_n - [b + f(t_n)]|^2 + \lambda \|f\|_{\mathcal{H}_k}^2 \right]$$
s.t.
$$v_{\min} \leq f'(t), \quad \forall t \in \mathcal{T}.$$

Task-2b: convoy localization, multiple vehicles ($Q \ge 1$)

- Data: $\left\{(t_{q,n},x_{q,n})_{n\in[N_q]}\right\}_{q\in[Q]}\subseteq\mathcal{T}\times\mathbb{R}.$
- Constraints: speed (v_{\min}) , inter-vehicular distance (d_{\min}) .
- Objective:

$$\min_{\substack{f_1, \dots, f_Q \in \mathcal{H}_k, \\ b_1, \dots, b_Q \in \mathbb{R}}} \frac{1}{Q} \sum_{q=1}^{Q} \left[\left(\frac{1}{N_q} \sum_{n=1}^{N_q} |x_{q,n} - (b_q + f_q(t_{q,n}))|^2 \right) + \lambda ||f_q||_{\mathcal{H}_k}^2 \right]$$
s.t.

$$egin{aligned} d_{\mathsf{min}} + b_{q+1} + f_{q+1}(t) &\leq b_q + f_q(t), orall q \in [Q-1], \ t \in \mathcal{T}, \ & \mathbf{v}_{\mathsf{min}} &\leq f_q'(t), & orall q \in [Q], \ t \in \mathcal{T}. \end{aligned}$$

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s.t.
$$\mathbf{d}_{\min} + b_{q+1} + f_{q+1}(t) \leq b_q + f_q(t), \forall q \in [Q-1], \ t \in \mathcal{T},$$

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Constraints

function values (f_q) and derivatives $(f_q^{'})$ with interaction $(f_q - f_{q+1})$, bias terms (b_q) with interaction $(b_{q+1} - b_q)$.

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- Initial condition: z(0) = 0 and $\dot{z}(0) = 0$.
- Control task (LQ = linear dynamics & quadratic cost):

$$\begin{aligned} & \min_{u \in L^2(\mathcal{T}, \mathbb{R})} \quad \int_{\mathcal{T}} |u(t)|^2 \mathrm{d}t \\ & \text{s.t.} \\ & z(0) = 0, \quad \dot{z}(0) = 0, \\ & \ddot{z}(t) = -\dot{z}(t) + u(t), \ \forall t \in \mathcal{T}, \\ & z_{\mathsf{low}}(t) \leq z(t) \leq z_{\mathsf{up}}(t), \ \forall \ t \in \mathcal{T}. \end{aligned}$$

Task-3: safety-critical control – continued

• With full state $\mathbf{f}(t) := [z(t); \dot{z}(t)] \in \mathbb{R}^2$

$$\dot{\mathbf{f}}(t) = \mathbf{A}\mathbf{f}(t) + \mathbf{B}u(t), \, \mathbf{f}(0) = \mathbf{0}, \, \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$

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 \bullet The controlled trajectories f belong to a $\mathbb{R}^2\text{-valued}$ RKHS with kernel

$$k(s,t) := \int_0^{\min(s,t)} e^{(s-\tau)\mathbf{A}} \mathbf{B} \mathbf{B}^{\top} e^{(t-\tau)\mathbf{A}^{\top}} d\tau, \quad s,t \in \mathcal{T},$$

and the task is

$$\begin{aligned} & \min_{\mathbf{f} = [f_1; f_2] \in \mathcal{H}_k} & \|\mathbf{f}\|_{\mathcal{H}_k}^2 \\ & \text{s.t.} \\ & z_{\mathsf{low}}(t) \leq f_1(t) \leq z_{\mathsf{up}}(t), \, \forall \, t \in \mathcal{T}. \end{aligned}$$

Task-3: safety-critical control – finished

- Assume for simplicity: z_{low} and z_{up} are piece-wise constant.
- Task:

$$\begin{aligned} & \min_{\mathbf{f} = [f_1; f_2] \in \mathcal{H}_k} & \|\mathbf{f}\|_{\mathcal{H}_k}^2 \\ & \text{s.t.} \\ & z_{\mathsf{low}, m} \leq f_1(t) \leq z_{\mathsf{up}, m}, \ \forall \ t \in \mathcal{T}_m, \ \forall m \in [M]. \end{aligned}$$

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Constraints

linear transformation of functions (f_1) , with matrix-valued kernel.

$$\begin{split} \left(\overline{\mathbf{f}}, \overline{\mathbf{b}}\right) &= \underset{\substack{\mathbf{f} = (f_q)_{q \in [Q]} \in (\mathcal{H}_k)^Q, \\ \mathbf{b} = (b_q)_{q \in [Q]} \in \mathcal{B}, \\ (\mathbf{f}, \mathbf{b}) \in \mathcal{C}}} \\ \end{split}$$

$$\begin{split} \left(\overline{\mathbf{f}}, \overline{\mathbf{b}}\right) &= \underset{\mathbf{f} = (f_q)_q \in [Q]}{\text{arg min}} & \mathcal{L}(\mathbf{f}, \mathbf{b}), \\ \mathbf{f} &= (f_q)_q \in [Q]} \in \mathcal{B}, \\ & \mathbf{b} = (b_q)_q \in [Q]} \in \mathcal{B}, \\ & (\mathbf{f}, \mathbf{b}) \in \mathcal{C} \end{split}$$

$$\mathcal{L}(\mathbf{f}, \mathbf{b}) &= L\left(\mathbf{b}, \left(\mathbf{x}_n, y_n, (f_q(\mathbf{x}_n))_{q \in [Q]}\right)_{n \in [N]}\right) + \Omega\left(\left(\|f_q\|_{\mathcal{H}_k}\right)_{q \in [Q]}\right), \end{split}$$

$$\begin{split} \left(\overline{\mathbf{f}}, \overline{\mathbf{b}}\right) &= \underset{\mathbf{f} = (f_q)_{q \in [Q]} \in (\mathcal{H}_k)^Q, \\ \mathbf{b} = (b_q)_{q \in [Q]} \in \mathcal{B}, \\ (\mathbf{f}, \mathbf{b}) &\in \mathcal{C} \end{split}$$

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$$\mathcal{C} &= \left\{\left(\mathbf{f}, \mathbf{b}\right) \mid \left(\mathbf{b}_0 - \mathbf{U}\mathbf{b}\right)_i \leq D_i(\mathbf{W}\mathbf{f} - \mathbf{f}_0)_i(\mathbf{x}), \quad \forall \mathbf{x} \in K_i, \forall i \in [I]\right\}, \end{split}$$

$$\begin{split} \left(\overline{\mathbf{f}}, \overline{\mathbf{b}}\right) &= \underset{\mathbf{f} = (f_q)_{q \in [Q]} \in (\mathcal{H}_k)^Q, \\ \mathbf{b} = (b_q)_{q \in [Q]} \in \mathcal{B}, \\ (\mathbf{f}, \mathbf{b}) &\in \mathcal{C} \end{split}$$

$$\mathcal{L}(\mathbf{f}, \mathbf{b}) = L\left(\mathbf{b}, \left(\mathbf{x}_n, y_n, (f_q(\mathbf{x}_n))_{q \in [Q]}\right)_{n \in [N]}\right) + \Omega\left(\left(\|f_q\|_{\mathcal{H}_k}\right)_{q \in [Q]}\right), \\ \mathcal{C} &= \left\{\left(\mathbf{f}, \mathbf{b}\right) \mid (\mathbf{b}_0 - \mathbf{U}\mathbf{b})_i \leq D_i(\mathbf{W}\mathbf{f} - \mathbf{f}_0)_i(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{K}_i, \forall i \in [I]\right\}, \\ (\mathbf{W}\mathbf{f})_i &= \sum_{q \in [Q]} W_{i,q} f_q, \end{split}$$

 $i \in [n_{i}]$

$$\begin{split} \left(\overline{\mathbf{f}}, \overline{\mathbf{b}}\right) &= \underset{\mathbf{f} = (f_q)_{q \in [Q]} \in (\mathcal{H}_k)^Q, \\ \mathbf{b} = (b_q)_{q \in [Q]} \in \mathcal{B}, \\ (\mathbf{f}, \mathbf{b}) &\in \mathcal{C} \end{split}$$

$$\mathcal{L}(\mathbf{f}, \mathbf{b}) = L\left(\mathbf{b}, \left(\mathbf{x}_n, y_n, (f_q(\mathbf{x}_n))_{q \in [Q]}\right)_{n \in [N]}\right) + \Omega\left(\left(\|f_q\|_{\mathcal{H}_k}\right)_{q \in [Q]}\right), \\ \mathcal{C} &= \left\{\left(\mathbf{f}, \mathbf{b}\right) \mid \left(\mathbf{b}_0 - \mathbf{U}\mathbf{b}\right)_i \leq D_i(\mathbf{W}\mathbf{f} - \mathbf{f}_0)_i(\mathbf{x}), \quad \forall \mathbf{x} \in K_i, \forall i \in [I]\right\}, \\ \left(\mathbf{W}\mathbf{f}\right)_i &= \sum_{q \in [Q]} W_{i,q} f_q, \\ D_i &= \sum_{q \in [Q]} \gamma_{i,j} \partial^{\mathbf{r}_{i,j}}, \ |\mathbf{r}_{i,j}| \leq s, \ \gamma_{i,j} \in \mathbb{R}, \ \partial^{\mathbf{r}} f(\mathbf{x}) = \frac{\partial^{|\mathbf{r}|} f(\mathbf{x})}{\partial_{x_1}^{x_1} \cdots \partial_{x_d}^{x_d}}. \end{split}$$

Blanket assumptions

- **1** Domain $\mathfrak{X} \subseteq \mathbb{R}^d$: open. Kernel $k \in \mathcal{C}^s(\mathfrak{X} \times \mathfrak{X})$.
- **2** $K_i \subset \mathfrak{X}$: compact, $\forall i$.
- **3** $\mathbf{f}_{0,i} \in \mathcal{H}_k$ for $\forall i$.
- **4** Bias domain $\mathcal{B} \subseteq \mathbb{R}^Q$: convex.
- **5** Loss L restricted to \mathcal{B} : strictly convex in \mathbf{b} .
- **6** Regularizer Ω : strictly increasing in each of its argument.

Our strenghtened SOC-constrained formulation

$$(\mathbf{f}_{\eta}, \mathbf{b}_{\eta}) = \underset{\mathbf{f} \in (\mathcal{H}_{k})^{Q}, \mathbf{b} \in \mathcal{B}}{\operatorname{arg min}} \mathcal{L}(\mathbf{f}, \mathbf{b})$$

$$\operatorname{s.t.}$$

$$(\mathbf{b}_{0} - \mathbf{U}\mathbf{b})_{i} + \underset{\eta_{i}}{\eta_{i}} \|(\mathbf{W}\mathbf{f} - \mathbf{f}_{0})_{i}\|_{\mathcal{H}_{k}}$$

$$\leq \min_{m \in [M_{i}]} D_{i}(\mathbf{W}\mathbf{f} - \mathbf{f}_{0})_{i}(\tilde{\mathbf{x}}_{i,m}), \ \forall i \in [I],$$

$$(\mathfrak{C}_{\eta})$$

where

- $\{\tilde{\mathbf{x}}_{i,m}\}_{m\in[M_i]}$: a δ_i -net of K_i in $\|\cdot\|_{\mathfrak{X}}$,
- $\bullet \ \eta_i = \sup_{m \in [M_i], \mathbf{u} \in \mathbb{B}_{\|\cdot\|_{\mathcal{X}}}(\mathbf{0}, \mathbf{1})} \|D_{i, \mathbf{x}} k(\tilde{\mathbf{x}}_{i, m}, \cdot) D_{i, \mathbf{x}} k(\tilde{\mathbf{x}}_{i, m} + \delta_i \mathbf{u}, \cdot)\|_{\mathcal{H}_k},$
- $D_{i,\mathbf{x}}k(\mathbf{x}_0,\cdot) := \mathbf{y} \mapsto D_i(\mathbf{x} \mapsto k(\mathbf{x},\mathbf{y}))(\mathbf{x}_0).$

Let
$$s = 0$$
, $I = 1$. Recall constraint (\mathcal{C}):
$$\{(\mathbf{f}, \mathbf{b}) \mid \underbrace{(b_0 - \mathbf{U}\mathbf{b})}_{\beta} \leq \underbrace{(\mathbf{W}\mathbf{f} - f_0)}_{\phi}(\mathbf{x}), \quad \forall \mathbf{x} \in K\}$$

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$$\Phi(K) := \{k(\mathbf{x}, \cdot) : \mathbf{x} \in K\} \subseteq H_{\phi, \beta}^+ := \{g \in \mathcal{H}_k \mid \beta \leq \langle \phi, g \rangle_{\mathcal{H}_k}\}$$

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• (\mathcal{C}_{η}) means: covering of $\Phi(K)$ by balls with η -radius centered at the $k\left(\tilde{\mathbf{x}}_{m},\cdot\right)$ is in the halfspace $H_{\phi,\beta}^{+}$; hence it is tightening.

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- (\mathcal{C}_{η}) means: covering of $\Phi(K)$ by balls with η -radius centered at the $k(\tilde{\mathbf{x}}_m, \cdot)$ is in the halfspace $H_{\phi,\beta}^+$; hence it is tightening.
- ullet η is obtained as the minimal radius.

Theorem

- Minimal values: $v_{\text{disc}} = \text{value of } (\mathcal{P}_{\eta}) \text{ with '} \eta = \mathbf{0}', \ \bar{\mathbf{v}} = \mathcal{L}\left(\bar{\mathbf{f}}, \bar{\mathbf{b}}\right), \ v_{\eta} = \mathcal{L}\left(\mathbf{f}_{\eta}, \mathbf{b}_{\eta}\right).$
- Let $\mathbf{f}_{\eta} = (f_{\eta,q})_{q \in [Q]}$.

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- Let $\mathbf{f}_{\eta} = (f_{\eta,q})_{q \in [Q]}$.

Then,

ullet (i) Tightening: any (\mathbf{f},\mathbf{b}) satisfying (\mathcal{C}_{η}) also satisfies (\mathcal{C}) , hence

$$v_{\rm disc} \leq \bar{\mathbf{v}} \leq v_{\boldsymbol{\eta}}.$$

Theorem

- Minimal values: $v_{\text{disc}} = \text{value of } (\mathfrak{P}_{\eta}) \text{ with } '\eta = \mathbf{0}', \ \bar{v} = \mathcal{L}\left(\bar{\mathbf{f}}, \bar{\mathbf{b}}\right), \ v_{\eta} = \mathcal{L}\left(\mathbf{f}_{\eta}, \mathbf{b}_{\eta}\right).$
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ullet (i) Tightening: any (\mathbf{f}, \mathbf{b}) satisfying (\mathcal{C}_{η}) also satisfies (\mathcal{C}) , hence

$$v_{\rm disc} \leq \overline{v} \leq v_{\eta}$$
.

• (ii) Representer theorem: For $\forall q \in [Q]$, $\exists \tilde{a}_{i,0,q}, \tilde{a}_{i,m,q}, a_{n,q} \in \mathbb{R}$ s.t.

$$f_{\eta,q} = \sum_{i \in [I]} \left[\tilde{\mathbf{a}}_{i,0,q} f_{0,i} + \sum_{m \in [M_i]} \tilde{\mathbf{a}}_{i,m,q} D_{i,\mathbf{x}} k \left(\tilde{\mathbf{x}}_{i,m}, \cdot \right) \right] + \sum_{n \in [N]} \mathbf{a}_{n,q} k(\mathbf{x}_n, \cdot).$$

Theorem – continued

• (iii) Performance guarantee: if \mathcal{L} is $(\mu_{f_q}, \mu_{\mathbf{b}})$ -strongly convex w.r.t. (f_q, \mathbf{b}) for any $q \in [Q]$, then

$$\|f_{\boldsymbol{\eta},q} - \bar{f}_q\|_{\mathfrak{H}_k} \leq \sqrt{\frac{2(\boldsymbol{v_{\boldsymbol{\eta}}} - \boldsymbol{v_{\mathsf{disc}}})}{\mu_{f_q}}}, \quad \|\mathbf{b_{\boldsymbol{\eta}}} - \bar{\mathbf{b}}\|_2 \leq \sqrt{\frac{2(\boldsymbol{v_{\boldsymbol{\eta}}} - \boldsymbol{v_{\mathsf{disc}}})}{\mu_{\mathbf{b}}}}.$$

Theorem - continued

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If in addition \mathbf{U} is surjective, $\mathcal{B} = \mathbb{R}^Q$, and $\mathcal{L}(\mathbf{\bar{f}}, \cdot)$ is L_b -Lipschitz continuous on $\mathbb{B}_{\|\cdot\|_2}\left(\mathbf{\bar{b}}, c_f \|\boldsymbol{\eta}\|_{\infty}\right)$ where

$$c_f = \sqrt{d} \left\| \left(\mathbf{U}^T \mathbf{U} \right)^{-1} \mathbf{U}^T \right\| \max_{i \in [I]} \left\| (\mathbf{W} \overline{\mathbf{f}} - \mathbf{f}_0)_i \right\|_{\mathcal{H}_k}$$
, then

$$\|f_{\boldsymbol{\eta},q} - \bar{f}_q\|_{\mathfrak{H}_k} \leq \sqrt{\frac{2L_b c_f \|\boldsymbol{\eta}\|_{\infty}}{\mu_{f_q}}}, \|\mathbf{b}_{\boldsymbol{\eta}} - \bar{\mathbf{b}}\|_2 \leq \sqrt{\frac{2L_b c_f \|\boldsymbol{\eta}\|_{\infty}}{\mu_{\mathbf{b}}}}.$$

Theorem - continued

• (iii) Performance guarantee: if \mathcal{L} is $(\mu_{f_q}, \mu_{\mathbf{b}})$ -strongly convex w.r.t. (f_q, \mathbf{b}) for any $q \in [Q]$, then

$$\|f_{\boldsymbol{\eta},\boldsymbol{q}} - \bar{f}_{\boldsymbol{q}}\|_{\mathfrak{H}_k} \leq \sqrt{\frac{2(\boldsymbol{v}_{\boldsymbol{\eta}} - \boldsymbol{v}_{\mathsf{disc}})}{\mu_{f_{\boldsymbol{q}}}}}, \quad \|\mathbf{b}_{\boldsymbol{\eta}} - \bar{\mathbf{b}}\|_2 \leq \sqrt{\frac{2(\boldsymbol{v}_{\boldsymbol{\eta}} - \boldsymbol{v}_{\mathsf{disc}})}{\mu_{\mathbf{b}}}}.$$

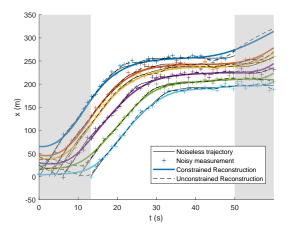
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$$\|f_{\boldsymbol{\eta},q} - \bar{f}_q\|_{\mathfrak{H}_k} \leq \sqrt{\frac{2L_b c_f \|\boldsymbol{\eta}\|_{\infty}}{\mu_{f_q}}}, \|\mathbf{b}_{\boldsymbol{\eta}} - \bar{\mathbf{b}}\|_2 \leq \sqrt{\frac{2L_b c_f \|\boldsymbol{\eta}\|_{\infty}}{\mu_{\mathbf{b}}}}.$$

1st bound: computable. 2nd: Larger $M_i \Rightarrow$ smaller $\delta_i \Rightarrow$ smaller $\eta_i \Rightarrow$ tighter bound.

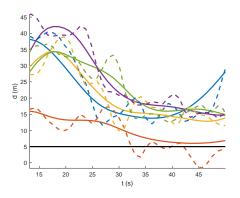
Demo (task-1): convoy localization with traffic jam

Setting: Q = 6, $d_{min} = 5m$, $v_{min} = 0$.



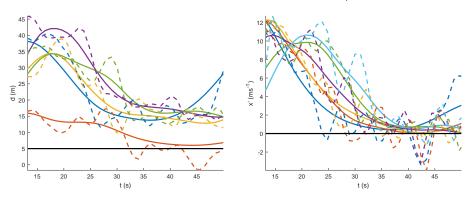
Demo (task-1): continued

Pairwise distances: $t\mapsto f_q(t)-f_{q+1}(t)$



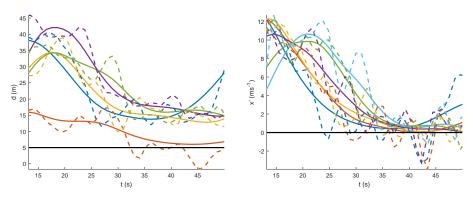
Demo (task-1): continued

Pairwise distances: $t\mapsto f_q(t)-f_{q+1}(t)$ Speed: $t\mapsto f_q'(t)$



Demo (task-1): continued

Pairwise distances: $t\mapsto f_q(t)-f_{q+1}(t)$ Speed: $t\mapsto f_q'(t)$

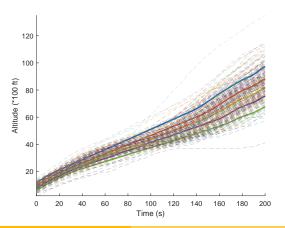


Shape constraints: especially relevant in noisy situations.

Demo (task-2): joint quantile regression

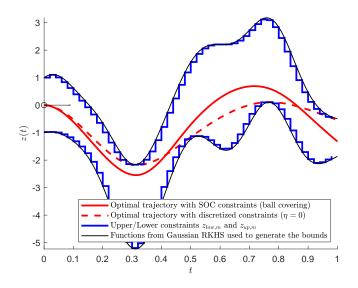
Analysis of aircraft trajectories, ENAC [Nicol, 2013]

- y: radar-measured altitude of aircrafts flying between two cities (Paris & Toulouse); x: time. d=1, N=15657.
- Demo: $\tau_q \in \{0.1, 0.3, 0.5, 0.7, 0.9\}.$
- Constraint: non-crossing, \nearrow (takeoff).



Demo (task-3): control of underwater vehicle

Vs discretization-based approach (which might crash):



Summary

- Focus: hard affine shape constraints on derivatives & RKHS.
- Proposed framework: SOC-based tightening.
- Applications:
 - convoy localization,
 - joint quantile regression: aircraft trajectories,
 - safety-critical control.

References & acknowledgements

- Transportation systems [Aubin-Frankowski et al., 2020].
- Control aspect [Aubin-Frankowski, 2020].
- Method:
 - dim(y) = 1: [Aubin-Frankowski and Szabó, 2020]. Code @ GitHub.
 - dim(y) ≥ 1 and SDP constraints (say joint convexity, production functions): [Aubin-Frankowski and Szabó, 2021].

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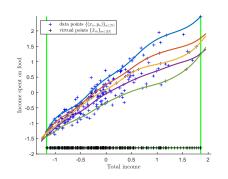


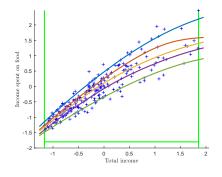
Demo (task-2): joint quantile regression

Economics:

- x: annual household income, y: food expenditure. d = 1, N = 235.
- Engel's law $\Rightarrow \nearrow$, concave.
- Demo: $\tau_q \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

Right: non-crossing, \nearrow , concave.





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