Rubik's on the Torus

Jeremy Alm, Michael Gramelspacher and Theodore Rice
(The American Mathematical Monthly, pp. 150-160, 2013)

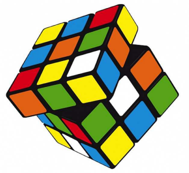
Zoltán Szabó

Gatsby Unit, Tea Talk

February 20, 2014

Outline

- Group: definitions.
- Rubik's Slide:
  1. task,
  2. solvability questions using groups.



Group: Definition

$(G \neq \emptyset, \cdot)$ is a group if

1. Associativity:
$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad (\forall a, b, c \in G). \tag{1}$$
2. Identity element: $\exists e \in G$ such that
$$e \cdot g = g \quad (\forall g \in G). \tag{2}$$
3. Inverse element: For $\forall g \in G$ $\exists g' \in G$ such that
$$g' \cdot g = e. \tag{3}$$

Abelian group: in addition $g_1 \cdot g_2 = g_2 \cdot g_1$ $(\forall g_1, g_2 \in G)$.

Examples

- Group:
    - $(\mathbb{R} \setminus \{0\}, \cdot)$, $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$.

Examples

- Group:
  - $(\mathbb{R} \setminus \{0\}, \cdot)$, $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$.
  - More generally ($\mathbb{F}$: field)
    - $(\mathbb{F} \setminus \{0\}, \cdot)$, $GL(\mathbb{F}, n) = (\{M \in \mathbb{F}^{n \times n} : \det(M) \neq 0\}, \cdot)$,
    - $(\mathbb{F}, +)$, $(\mathbb{F}^{n \times m}, +)$.

Examples

- Group:
  - $(\mathbb{R} \setminus \{0\}, \cdot)$, $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$.
  - More generally ($\mathbb{F}$: field)
    - $(\mathbb{F} \setminus \{0\}, \cdot)$, $GL(\mathbb{F}, n) = (\{M \in \mathbb{F}^{n \times n} : \det(M) \neq 0\}, \cdot)$,
    - $(\mathbb{F}, +)$, $(\mathbb{F}^{n \times m}, +)$.
  - $(S_X, \circ)$: bijections of set X with composition.
    - Example: $S_n := S_{\{1,\ldots,n\}}$ = permutations of $\{1, \ldots, n\}$ – symmetric group.

Examples

- Group:
  - $(\mathbb{R} \setminus \{0\}, \cdot)$, $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$.
  - More generally ($\mathbb{F}$: field)
    - $(\mathbb{F} \setminus \{0\}, \cdot)$, $GL(\mathbb{F}, n) = (\{M \in \mathbb{F}^{n \times n} : \det(M) \neq 0\}, \cdot)$,
    - $(\mathbb{F}, +)$, $(\mathbb{F}^{n \times m}, +)$.
  - $(S_X, \circ)$: bijections of set X with composition.
    - Example: $S_n := S_{\{1,\ldots,n\}}$ = permutations of $\{1, \ldots, n\}$ – symmetric group.
- *Non*-group: $(\mathbb{R}, \cdot)$ – $\nexists 0^{-1}$.

Subgroup

- Definition: $H \leq (G, \cdot)$ is a subgroup of G if
  - informally: it is a group with the operation inherited from G.
  - formally: $H \subseteq G$ and
    - $e \in H$.
    - $\forall g \in H$: $g^{-1} \in H$.
    - $\forall g_1, g_2 \in H$: $g_1 \cdot g_2 \in H$.
- Example: $(\mathbb{Z}, +) \leq (\mathbb{R}, +)$.
- *Non*-example: $(\mathbb{R} \setminus \{0\}, \cdot) \not\leq (\mathbb{R}, +)$ – different operations.

Generated Subgroup

- The intersection of groups is group. $\Rightarrow$
- $\exists$ generated subgroup: $g_1, \ldots, g_n \in G$

$$\langle g_1, \ldots, g_n \rangle = \bigcap_{g_1, \ldots, g_n \in H \leq G} H. \tag{4}$$

- Explicit formula:

$$\langle g_1, \ldots, g_n \rangle = \Big\{ g_{i_1}^{\epsilon_1} \cdot \ldots \cdot g_{i_k}^{\epsilon_k} : 1 \leq i_1 \leq \ldots \leq i_k \leq n, \\ \epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}, k \geq 0 \Big\}. \tag{5}$$

Left/Right Cosets

- Given: $H \leq G$. Left/right cosets of H containing $g \in G$ are

$$gH = \{gh : h \in H\}, \qquad Hg = \{hg : h \in H\}. \tag{6}$$

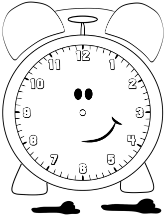
- Property: the cosets of $H \leq G$ form a *partition* of G.

Left/Right Cosets

- Given: $H \leq G$. Left/right cosets of H containing $g \in G$ are

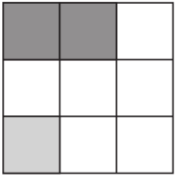
$$gH = \{gh : h \in H\}, \qquad Hg = \{hg : h \in H\}. \tag{6}$$

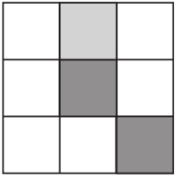
- Property: the cosets of $H \leq G$ form a *partition* of G.
- Example: hours on a clock [$(12\mathbb{Z}, +) \leq (\mathbb{Z}, +)$].



Rubik's Slide: Task

- Initial state, final state:

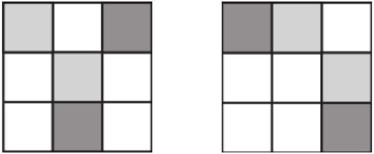




- Allowed moves $\in S_9$:
  - shift by one space (up/down/left/right),
  - rotation of border squares (clock/counter clockwise).

Move: Shift to the Right – h(orizontal)

- Shift to the right:



- Position changes:

1	2	3
4	5	6
7	8	9

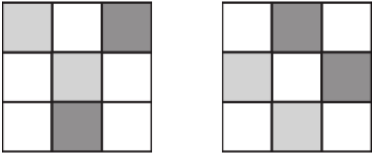
$\longmapsto$

3	1	2
6	4	5
9	7	8

- $h = (1,2,3)(4,5,6)(7,8,9) \in S_9$ [cycle notation].

Move: Shift Down – v(ertical)

- Shift down:



- Position changes:

1	2	3
4	5	6
7	8	9

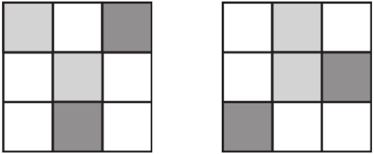
$\longmapsto$

7	8	9
1	2	3
4	5	6

- $v = (1,4,7)(2,5,8)(3,6,9) \in S_9$.

Move: Clockwise Rotation – c(lockwise)

- Clockwise rotation:



- Position changes:

1	2	3
4	5	6
7	8	9

$\longmapsto$

4	1	2
7	5	3
8	9	6

- $c = (1, 2, 3, 6, 9, 8, 7, 4) \in S_9$.

Rubik's Slide: Questions

- Goal: initial state $\rightarrow$ final state by allowed moves. Example:

$$h^{-1}vc^3. \tag{7}$$

- Questions:
  1. Can any (initial,final) state pair be solved, i.e.,

$$\langle h, v, c\rangle = S_9? \tag{8}$$

  2. Practical solutions?

Lemma

- $\langle h, v \rangle$: Abelian since

$$hv = (1,5,9)(2,6,7)(3,4,8) = vh \tag{9}$$

  and $h^3 = v^3 = e$. Thus, $\langle h, v \rangle = \{h^i v^j : i, j = 0, 1, 2\}$.
- $c^3 h = (1,4)(2,7,3,9,5,6,8) \Rightarrow (c^3 h)^7 = (1,4)$.

$\langle h, v, c \rangle$ contains every transposition

Transpose two squares x and y:

1. σ: h/v-s to move x to Position 5; then
2. τ: c-s to rotate y to Position 2.
3. Apply h^{-1}: $(x, y) \Rightarrow$ Position $(5, 2) \rightarrow (4, 1)$.
4. Apply $(c^3 h)^7 = (1, 4)$: swaps x and y. Finally, $\left(\sigma\tau h^{-1}\right)^{-1}$.

Transposition $(x, y) = \left(\sigma\tau h^{-1}\right)(c^3 h)^7 \left(\sigma\tau h^{-1}\right)^{-1}$.

1	2	3
4	5	6
7	8	9

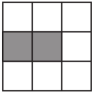
Consequence = Answer$_1$

- Transposition $(x, y) \in \langle h, v, c \rangle \Rightarrow \langle h, v, c \rangle = S_9$.
- Specially: every square can have different color.
- Since

$$v = c^2 h^2 c^{-2} \tag{10}$$

$\langle h, c \rangle = \langle h, v, c \rangle = S_9$.

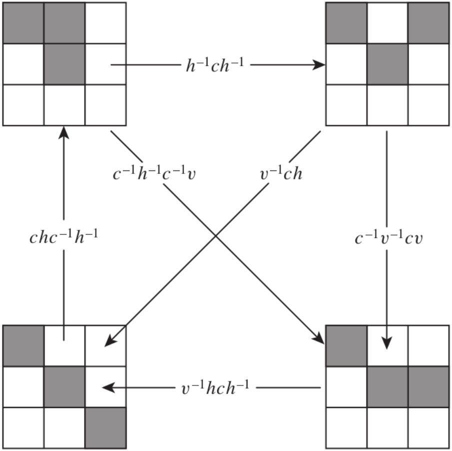
Answer_2 = Practical Solution



- Assumption: only one (non-white) color is present.
- Number of colored squares = 1 or 2: simple.
- $=$ 3: wheel of 4.
- $=$ 4: wheel of 7 (similar to the previous case).

The Wheel of 4

Consider all states, where the center is colored; form equivalence classes by rotations. [$\approx$ cosets of $H = \{e, c, c^2, c^3\}$]



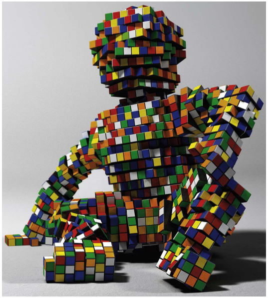
Solution Using the Wheel of 4: $S_0 \to S_\omega$

1. $\sigma : S_0 \to S_1 \to S_2$:
   1. h or v so that the middle square is occupied,
   2. c-s to match a state on the wheel.
2. $\tau : S_\omega \to S_{\omega-1} \to S_{\omega-2}$: similarly to the previous step.
3. w : Use the wheel to move S_2 to $S_{\omega-2}$.

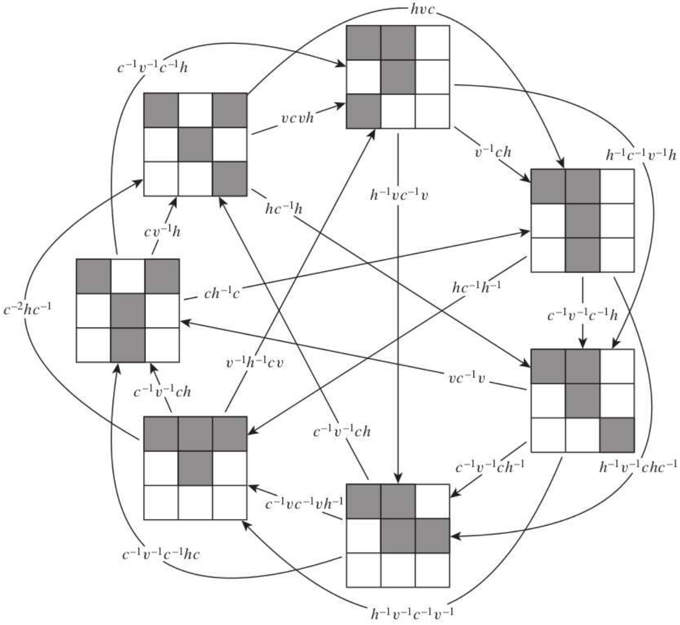
$$S_0 \xrightarrow{\sigma} \boxed{S_2 \xrightarrow{w} S_{\omega-2}} \xleftarrow{\tau} S_\omega. \tag{11}$$

Summary

- Constraint satisfaction problem: Rubik's slide.
- Constructive solutions using groups.
- Rubik's slide: $\exists$ app.



Thank you for the attention!



hvc
c⁻¹v⁻¹c⁻¹h
vcvh
v⁻¹ch
h⁻¹c⁻¹v⁻¹h
hc⁻¹h
h⁻¹vc⁻¹v
cv⁻¹h
ch⁻¹c
hc⁻¹h⁻¹
c⁻²hc⁻¹
c⁻¹v⁻¹c⁻¹h
v⁻¹h⁻¹cv
vc⁻¹v
c⁻¹v⁻¹ch
c⁻¹v⁻¹ch
c⁻¹v⁻¹ch⁻¹
h⁻¹v⁻¹chc⁻¹
c⁻¹vc⁻¹vh⁻¹
c⁻¹v⁻¹c⁻¹hc
h⁻¹v⁻¹c⁻¹v⁻¹