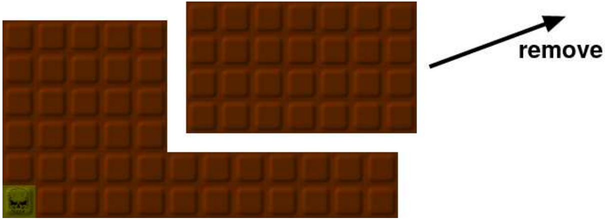
Nim & Friends

Zoltán Szabó

Gatsby Unit, Tea Talk
January 11, 2016

Ingredients:

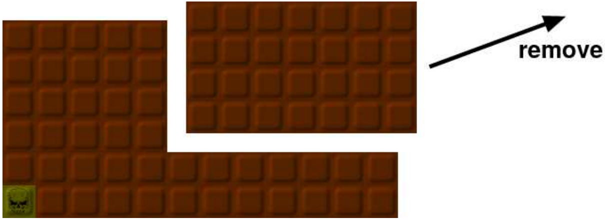
- 2 chocolate lovers,
- 1 bar of chocolate.



The poisoned chocolate game = Chomp

Ingredients:

- 2 chocolate lovers,
- 1 bar of chocolate.



URL: `https://www.math.ucla.edu/~tom/Games/chomp.html`

Nim: demo

Ingredients:

- 3 piles of stones.
- 2 players.

Nim: demo

Ingredients:

- 3 piles of stones.
- 2 players.

Winner: who takes the last stone(s).

Nim: demo

Ingredients:

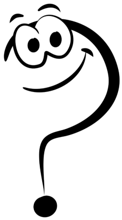
- 3 piles of stones.
- 2 players.

Winner: who takes the last stone(s).

URL: `https://www.dotsphinx.com/games/nim/`

Questions

- Does P(layer-)1 or P2 have optimal strategy?
- How should they play?



$(n_1, \ldots, n_k)$-Nim

- Pile sizes: $n_1, \ldots, n_k$ (demo: $n_1 = 1, n_2 = 2, n_3 = 3$).
- $a \oplus b$: Nim-sum ($\mathbb{Z}_2$, bitwise).
- In our demo:

$$\begin{array}{r}
01 \leftrightarrow n_1 \\
10 \leftrightarrow n_2 \\
\oplus 11 \leftrightarrow n_3 \\
\hline
00 \leftrightarrow n_1 \oplus n_2 \oplus n_3
\end{array}$$

$(n_1, \ldots, n_k)$-Nim

- Pile sizes: $n_1, \ldots, n_k$ (demo: $n_1 = 1, n_2 = 2, n_3 = 3$).
- $a \oplus b$: Nim-sum ($\mathbb{Z}_2$, bitwise).
- In our demo:

$$\begin{array}{l} 01 \leftrightarrow n_1 \\ 10 \leftrightarrow n_2 \\ \oplus 11 \leftrightarrow n_3 \\ \hline 00 \leftrightarrow n_1 \oplus n_2 \oplus n_3 \end{array}$$

- Bouton, 1901: P2 has winning strategy $\Leftrightarrow \oplus_{j=1}^{k} n_j = 0$.

Simple solutions: motivated (Nim)

Simplified problem: 2D, rook (top right)

- Assume: $n_1 = 2, n_2 = 3$. $(0, 0) \in$ L(ooser).
- $L \xrightarrow{\forall} W$, $W \xrightarrow{\exists} L$

2				
1				
0	L			
	0	1	2	3

Simplified problem: 2D, rook (top right)

- Assume: $n_1 = 2, n_2 = 3$. $(0, 0) \in$ L(ooser).
- $L \xrightarrow{\forall} W$, $W \xrightarrow{\exists} L$

2				
1				
0	L			
	0	1	2	3

2	W			
1	W			
0	L	W	W	W
	0	1	2	3

Simplified problem: 2D, rook (top right)

- Assume: $n_1 = 2, n_2 = 3$. $(0,0) \in$ L(ooser).
- $L \xrightarrow{\forall} W$, $W \xrightarrow{\exists} L$

2	W			
1	W			
0	L	W	W	W
	0	1	2	3

2	W			
1	W	L		
0	L	W	W	W
	0	1	2	3

Simplified problem: 2D, rook (top right)

- Assume: $n_1 = 2, n_2 = 3$. $(0,0) \in$ L(ooser).
- $L \xrightarrow{\forall} W$, $W \xrightarrow{\exists} L$

2	W			
1	W	L		
0	L	W	W	W
	0	1	2	3

2	W	W		
1	W	L	W	W
0	L	W	W	W
	0	1	2	3

Simplified problem: 2D, rook (top right)

- Assume: $n_1 = 2, n_2 = 3$. $(0,0) \in$ L(ooser).
- $L \xrightarrow{\forall} W$, $W \xrightarrow{\exists} L$

2	W	W		
1	W	L	W	W
0	L	W	W	W
	0	1	2	3

2	W	W	L	
1	W	L	W	W
0	L	W	W	W
	0	1	2	3

Simplified problem: 2D, rook (top right)

- Assume: $n_1 = 2, n_2 = 3$. $(0,0) \in$ L(ooser).
- $L \xrightarrow{\forall} W$, $W \xrightarrow{\exists} L$

2	W	W	L	
1	W	L	W	W
0	L	W	W	W
	0	1	2	3

2	W	W	L	W
1	W	L	W	W
0	L	W	W	W
	0	1	2	3

$$\begin{array}{r} 10 \\ \oplus 11 \\ \hline 01 \end{array}$$

$1 \neq 0 \Rightarrow$ P1 can win; 'Bouton, 1901' might hold.

Let us prove it!

- $L := \{(n_1, \ldots, n_k) : \oplus_{i=1}^{k} n_i = 0\}$.
- **Goal**: $L \xrightarrow{\forall} W$, $W \xrightarrow{\exists} L$.
- $L \xrightarrow{\forall} W$:
  - We are in L: $\oplus_{i=1}^{k} n_i = n_j \oplus n_{-j} = 0$.
  - This will not hold upon Δn_j.

Let us prove it!

- $L := \{(n_1, \ldots, n_k) : \oplus_{i=1}^{k} n_i = 0\}$.
- **Sub-goal**: $W \xrightarrow{\exists} L$:
  - t: largest bit where $\exists$ difference.
  - n_j: a pile whose t^{th} bit is 1 (odd).

$$\begin{array}{r}
XYZU0\ldots \leftrightarrow n_{-j} \\
\oplus XYZU1\ldots \leftrightarrow n_j \\
\hline
0\ldots01\ldots \leftrightarrow n_{-j} \oplus n_j
\end{array}$$

Let us prove it!

- $L := \{(n_1, \ldots, n_k) : \oplus_{i=1}^{k} n_i = 0\}$.
- **Sub-goal**: $W \xrightarrow{\exists} L$:
  - t: largest bit where $\exists$ difference.
  - n_j: a pile whose t^{th} bit is 1 (odd).

$$\begin{array}{r} XYZU0\ldots \leftrightarrow n_{-j} \\ \oplus XYZU1\ldots \leftrightarrow n_j \\ \hline 0\ldots01\ldots \leftrightarrow n_{-j} \oplus n_j \end{array}$$

  - Flip 1 to 0 in n_j, tail copy of n_{-j} to n_j $\Rightarrow$ $n_{-j} \oplus n_{-j} = 0$; we get to L.

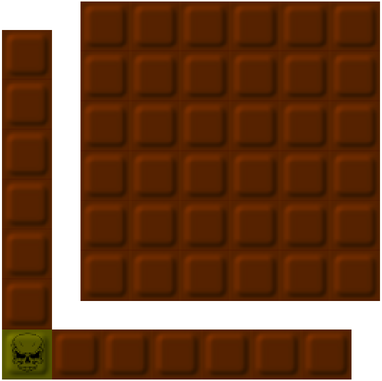
Chomp: 1^{st} player can win

Proof (strategy stealing):

- Assume the opposite: whole board $\in L \xrightarrow{\text{P1: } \forall} W$.
- P2 has winning strategy from P1's step.
- But P1 could have started there. ↯

Chomp: optimal solution?

- Square ($n \times n$):



- $2 \times n$: solved (longer).
- Rest: open.

Summary

- Combinatorial games:
  - Chomp (poisoned chocholate).
  - Nim (stone piles).
- Nim: ✓
- Chomp: existence.

State partitioning: W/L

- $N(p)$: length of the longest game from p.
- Induction on $N(p)$:
  - Sink(s) $\in L$.
  - New p: if

1. $p \xrightarrow{\text{possible}} q \in L \Rightarrow p \in W$.
2. $p \xrightarrow{\text{all}} q \in W \Rightarrow p \in L$.