Shape-Constrained Estimation in Reproducing Kernel Hilbert Spaces

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Joint work with:

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Palaiseau
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Outline

- Shape constraints.
- Kernels and RKHS-s.
- Task & results.

Shape constraints

Basic examples

1. Non-negativity:

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3. Convexity:

$$0 \leq f''(x) \quad (\forall x).$$

Other examples

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Example: generator of a d-variate Archimedean copula is $(d-2)$-alternating monotone.

Other examples from economics, game theory

6. Monotonicity w.r.t. partial ordering ($\mathbf{u} \preccurlyeq \mathbf{v} \Rightarrow f(\mathbf{u}) \leq f(\mathbf{v})$):

$\mathbf{u} \preccurlyeq \mathbf{v}$ iff

- $u_i \leq v_i$ ($\forall i$; product ordering),
- $\sum_{j \in [i]} u_j \leq \sum_{j \in [i]} v_j$ ($\forall i$; unordered weak majorization).

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6. Monotonicity w.r.t. partial ordering ($\mathbf{u} \preccurlyeq \mathbf{v} \Rightarrow f(\mathbf{u}) \leq f(\mathbf{v})$):

$$0 \leq \partial^{\mathbf{e}_j} f(\mathbf{x}), \quad (\forall j \in [d], \forall \mathbf{x}),$$

$$0 \leq \partial^{\mathbf{e}_d} f(\mathbf{x}) \leq \ldots \leq \partial^{\mathbf{e}_1} f(\mathbf{x}) \quad (\forall \mathbf{x}).$$

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7. Supermodularity:

$$0 \leq \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \quad (\forall i \neq j \in [d], \forall \mathbf{x}),$$

i.e. $f(\mathbf{u} \vee \mathbf{v}) + f(\mathbf{u} \wedge \mathbf{v}) \geq f(\mathbf{u}) + f(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$.

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Pattern

$$0 \leq Df(\mathbf{x}) \quad \forall \mathbf{x}.$$

Our high-level goal & challenge

- Given: $\{(\mathbf{x}_n, y_n)\}_{n\in[N]} \subset \mathbb{R}^d \times \mathbb{R}$ samples.
- Goal: find $f \in \mathcal{H}$ such that

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- soft constraints: finite many points.
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In our work

$\mathcal{H}$: RKHS; rich but tractable. Hard constraint & performance guarantees.

Kernel, RKHS

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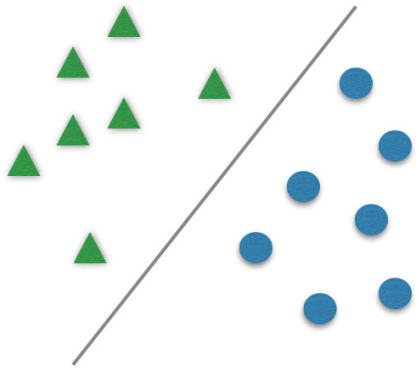
- Classification (SVM):



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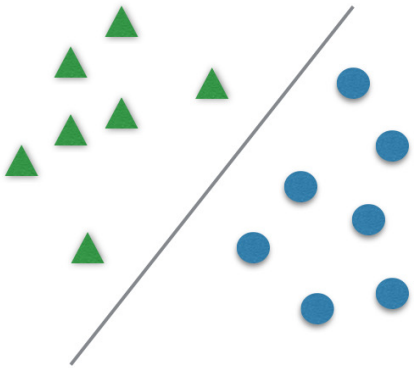

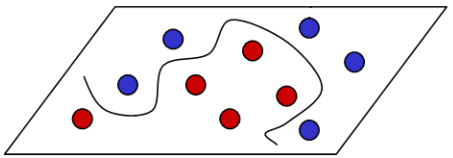

Input Space

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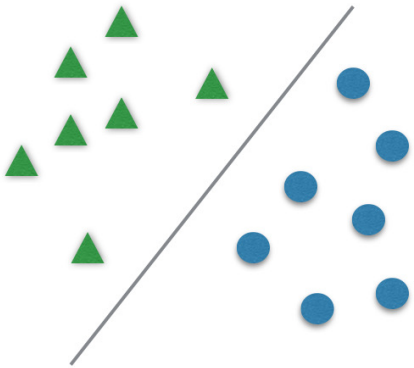

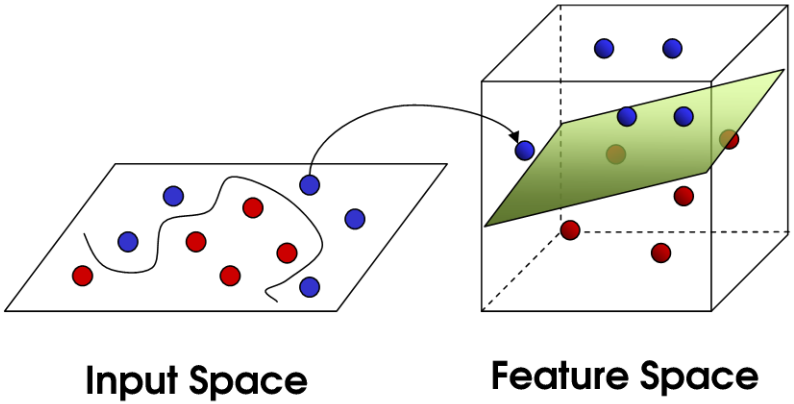

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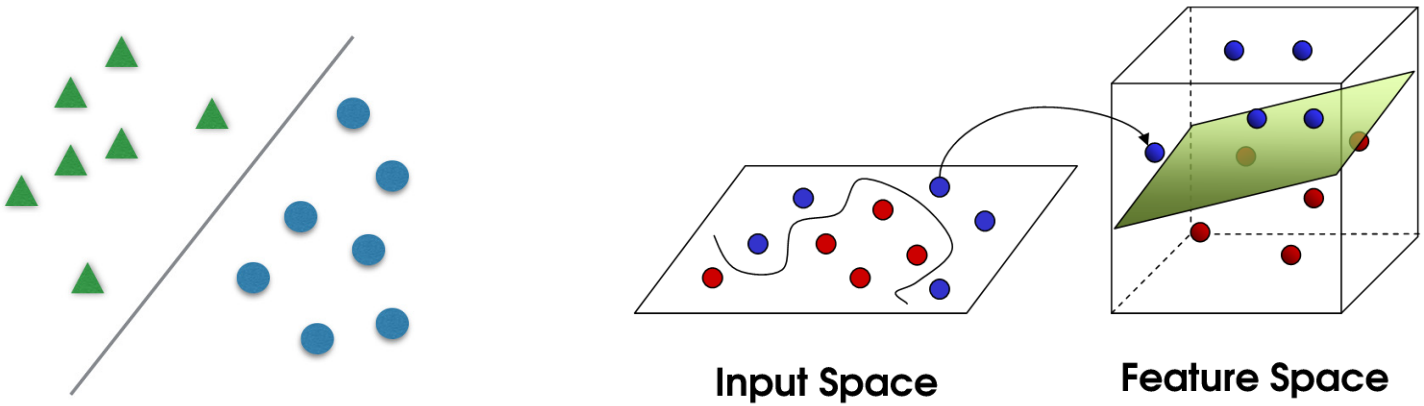

- Representation of distributions:

$$\mathbb{P} \mapsto \mathbb{E}_{\mathbf{x}\sim\mathbb{P}}\varphi(\mathbf{x}),$$

divergence measures (MMD), independence measures (HSIC, KCCA, KGV), hypothesis testing.

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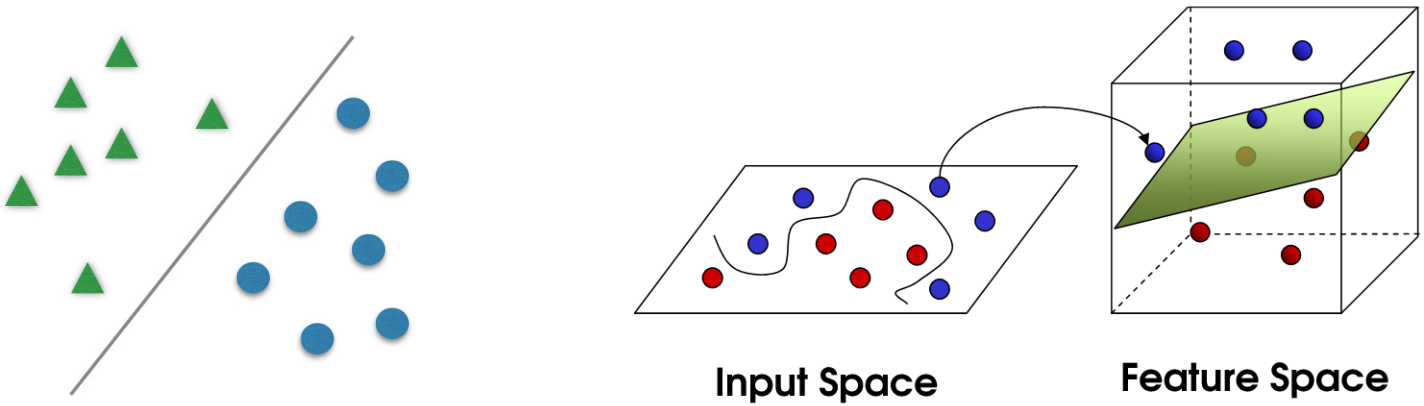

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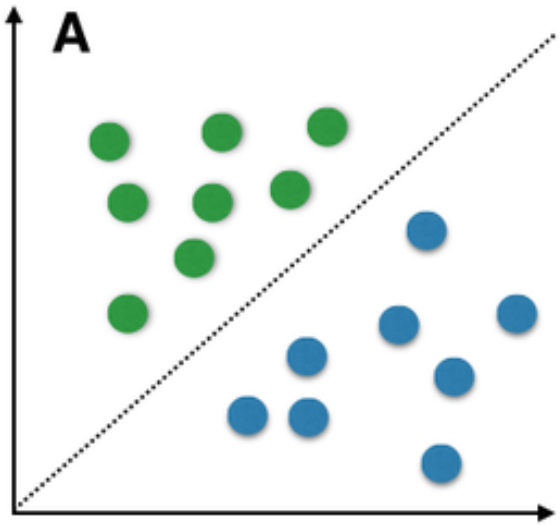
  divergence measures (MMD), independence measures (HSIC, KCCA, KGV), hypothesis testing.
- Gaussian processes (covariance function), Fourier analysis, ...

Classification motivation: linear separability

Idealized situation



Decision surface:

$$\{\mathbf{x} : \langle \mathbf{w}, \mathbf{x} \rangle = 0\}$$

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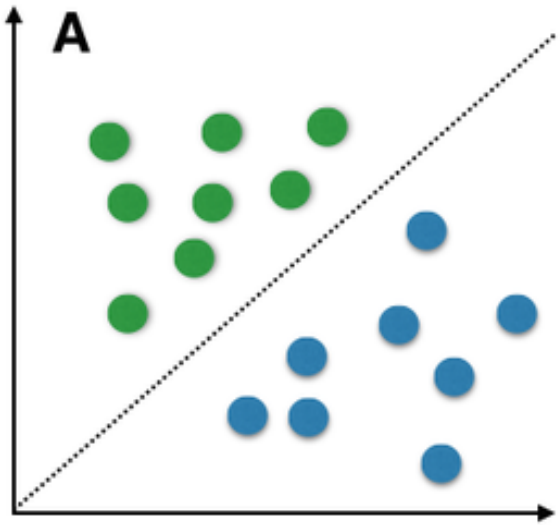

Decision surface:

$$\{\mathbf{x} : \langle \mathbf{w}, \mathbf{x} \rangle = 0\} \Rightarrow$$

classes:

$$\{\mathbf{x} : \langle \mathbf{w}, \mathbf{x} \rangle \geq 0\} \qquad \{\mathbf{x} : \langle \mathbf{w}, \mathbf{x} \rangle < 0\}$$

Classification motivation: non-linear separability

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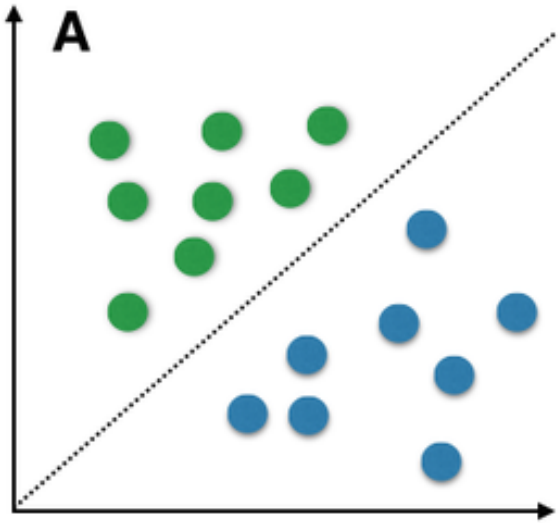
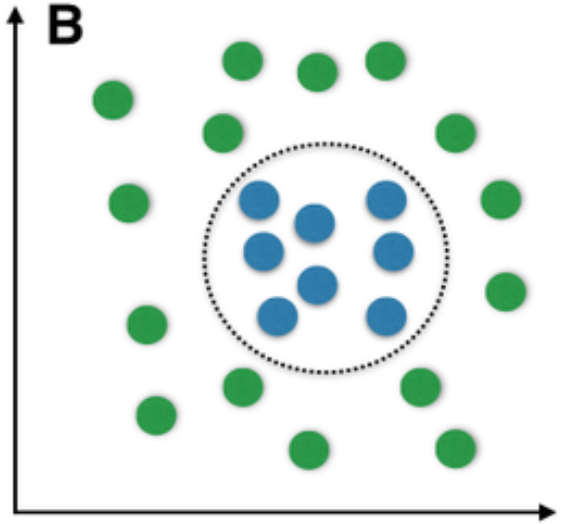


Real world



Decision surface (left):

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Non-linear separability – continued

On the ellipse

$$\left\{ \mathbf{x} : \frac{(x_1 - c_1)^2}{a^2} + \frac{(x_2 - c_2)^2}{b^2} = 1 \right\}$$

Non-linear separability – continued

On the ellipse, outside

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On the ellipse, outside, inside:

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With polynomial feature: $\varphi(\mathbf{x}) = \left(x_1^2, x_1, 1, x_2^2, x_2\right)$:

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Quadratic & polynomial features

Still in $\mathbb{R}^2$:

$$\varphi(\mathbf{x}) = \left(x_1^2, \sqrt{2}x_1x_2, x_2^2\right),$$

$\langle \varphi(\mathbf{x}), \varphi(\mathbf{x}') \rangle$ =?

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$\langle \mathbf{x}, \mathbf{x}' \rangle^d = \langle \varphi(\mathbf{x}), \varphi(\mathbf{x}') \rangle$: $\varphi(\mathbf{x}) = d$-order polynomial. $\Rightarrow$ Explicit computation would be heavy!

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- All these definitions are equivalent, $k \overset{1:1}{\leftrightarrow} \mathcal{H}_k$.
- $f(x) = \langle \mathbf{w}, \varphi(x) \rangle_{\mathbb{R}^M} = \sum_{m=1}^{M} w_m \varphi_m(x)$: Fourier analysis, splines, ...

Kernel examples on $\mathbb{R}^d$ ($\gamma, \sigma, \nu > 0$, $c \geq 0$, $p \in \mathbb{Z}^+$)

$$k_p(\mathbf{x}, \mathbf{y}) = (\langle \mathbf{x}, \mathbf{y} \rangle + c)^p$$

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$$k_p(\mathbf{x}, \mathbf{y}) = (\langle \mathbf{x}, \mathbf{y} \rangle + c)^p, \qquad k_G(\mathbf{x}, \mathbf{y}) = e^{-\gamma \|\mathbf{x}-\mathbf{y}\|_2^2},$$

$$k_e(\mathbf{x}, \mathbf{y}) = e^{-\gamma \|\mathbf{x}-\mathbf{y}\|_2}, \qquad k_L(\mathbf{x}, \mathbf{y}) = e^{-\gamma \|\mathbf{x}-\mathbf{y}\|_1},$$

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Kernel examples on $\mathbb{R}^d$ ($\gamma, \sigma, v > 0$, $c \geq 0$, $p \in \mathbb{Z}^+$)

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Or the flexible Matérn family:

$$k_M(\mathbf{x}, \mathbf{y}) = \frac{2^{1-v}}{\Gamma(v)} \left(\frac{\sqrt{2v}\, \|\mathbf{x} - \mathbf{y}\|_2}{\sigma} \right)^v K_v \left(\frac{\sqrt{2v}\, \|\mathbf{x} - \mathbf{y}\|_2}{\sigma} \right),$$

where

- K_v: modified Bessel function of the second kind of order v,
- Specific cases: For $v = \frac{1}{2}$ one gets $k(\mathbf{x}, \mathbf{y}) = e^{-\frac{\|\mathbf{x}-\mathbf{y}\|_2}{\sigma}}$. Gaussian kernel: $v \to \infty$.

Kernels on other domains ($\mathcal{X}$)

- Strings [Watkins, 1999, Lodhi et al., 2002, Leslie et al., 2002, Kuang et al., 2004, Leslie and Kuang, 2004, Saigo et al., 2004, Cuturi and Vert, 2005],
- time series [Rüping, 2001, Cuturi et al., 2007, Cuturi, 2011, Király and Oberhauser, 2019],
- trees [Collins and Duffy, 2001, Kashima and Koyanagi, 2002],
- groups and specifically rankings [Cuturi et al., 2005, Jiao and Vert, 2016],
- sets [Haussler, 1999, Gärtner et al., 2002],
- various generative models [Jaakkola and Haussler, 1999, Tsuda et al., 2002, Seeger, 2002, Jebara et al., 2004],
- fuzzy domains [Guevara et al., 2017], or
- graphs [Kondor and Lafferty, 2002, Gärtner et al., 2003, Kashima et al., 2003, Borgwardt and Kriegel, 2005, Shervashidze et al., 2009, Vishwanathan et al., 2010, Kondor and Pan, 2016, Bai et al., 2018].

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- k: bounded $[\sup_{x,y \in \mathcal{X}} k(x,y) \leq C] \Rightarrow \forall f \in \mathcal{H}_k$ is bounded

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- k: characteristic $\Leftrightarrow \mathbb{P} \mapsto \int_{\mathcal{X}} \varphi(x)\mathrm{d}\mathbb{P}(x) \in \mathcal{H}_k$ is injective.

Kernel puzzle

Let

$$\mathcal{X} = \{0, 1\},$$

$$k(x, x') = \left\{ \begin{array}{ll} 1, & \text{if } x \neq x' \\ -1, & \text{if } x = x' \end{array} \right\}.$$

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Easy-to-check conditions for a $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ function to be kernel ?

Kernel factory

Let k, k_m kernels, $\alpha, \alpha_m \in \mathbb{R}^{\geq 0}$. Then, the followings are also kernels:

1. Non-negative shift: $k + \alpha$.
2. Cone: $\sum_{m=1}^{M} \alpha_m k_m$.
3. Limit: $k(x, x') := \lim_{m \to \infty} k_m(x, x')$.
4. Pre-post multiplication: $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, $f : \mathcal{X} \to \mathbb{R}$

$$\tilde{k}(x, y) = f(x) k(x, y) f(y).$$

5. Product:

$$\left(\otimes_{m=1}^{M} k_m\right)\left((x_1, \ldots, x_M), \left(x'_1, \ldots, x'_M\right)\right) = \prod_{m=1}^{M} k_m\left(x_m, x'_m\right).$$

Task & Results

Task-1: convoy localization, one vehicle ($Q = 1$)

- Given: noisy time-location samples $\{(t_n, x_n)\}_{n\in[N]} \subset \underbrace{[0, T]}_{=:\mathcal{T}} \times \mathbb{R}$.
- Goal: learn the (t, x) relation.
- Constraint: lower bound on speed ($v_{\min}$).

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- Goal: learn the (t, x) relation.
- Constraint: lower bound on speed ($v_{\min}$).
- Objective:

$$\min_{b \in \mathbb{R},\, f \in \mathcal{H}_k} \left[\frac{1}{N} \sum_{n\in[N]} |x_n - (b + f(t_n))|^2 + \lambda \|f\|^2_{\mathcal{H}_k} \right]$$

$$\text{s.t.}$$

$$v_{\min} \leq f'(t), \quad \forall t \in \mathcal{T}.$$

Task-2: convoy localization, multiple vehicles ($Q \geq 1$)

- Data: $\left\{(t_{q,n}, x_{q,n})_{n\in[N_q]}\right\} \subseteq \mathcal{T} \times \mathbb{R}$, $q \in [Q]$.
- Constraints: speed ($v_{\min}$), inter-vehicular distance ($d_{\min}$).
- Objective:

$$\min_{\substack{f_1,\ldots,f_Q \in \mathcal{H}_k,\\ b_1,\ldots,b_Q \in \mathbb{R}}} \frac{1}{Q}\sum_{q=1}^{Q}\left[\left(\frac{1}{N_q}\sum_{n=1}^{N_q}|x_{q,n} - (b_q + f_q(t_{q,n}))\,|^2\right) + \lambda\|f_q\|^2_{\mathcal{H}_k}\right]$$

$$\text{s.t.}$$

$$d_{\min} + b_{q+1} + f_{q+1}(t) \leq b_q + f_q(t), \forall q \in [Q-1],\ t \in \mathcal{T},$$

$$v_{\min} \leq f_q'(t), \qquad \forall q \in [Q],\ t \in \mathcal{T}.$$

Task-2: joint quantile regression

- Given: $(\tau_q)_{q\in[Q]}$ levels $\nearrow$, $\{(\mathbf{x}_n, y_n)\}_{n\in[N]}$ samples.
- Estimate jointly the τ_q-quantiles of $\mathbb{P}(Y|X=\mathbf{x})$.

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$$\mathcal{L}(\mathbf{f}, \mathbf{b}) = \frac{1}{N} \sum_{q\in[Q]} \sum_{n\in[N]} l_{\tau_q}(y_n - [f_q(\mathbf{x}_n) + b_q]) + \lambda_{\mathbf{b}} \|\mathbf{b}\|_2^2 + \lambda_f \sum_{q\in[Q]} \|f_q\|_k^2,$$

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- Constraint (non-crossing): $K :=$ smallest rectangle containing $\{\mathbf{x}_n\}_{n\in[N]}$,

$$f_q(\mathbf{x}) + b_q \leq f_{q+1}(\mathbf{x}) + b_{q+1},\ \forall q \in [Q-1],\ \forall \mathbf{x} \in K.$$

Task: general

$$(\bar{\mathbf{f}}, \bar{\mathbf{b}}) = \underset{\substack{\mathbf{f}=(f_q)_{q\in[Q]} \in (\mathcal{H}_k)^Q, \\ \mathbf{b}=(b_q)_{q\in[Q]} \in \mathcal{B}, \\ (\mathbf{f},\mathbf{b}) \in C}}{\arg\min} \mathcal{L}(\mathbf{f}, \boldsymbol{b}),$$

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$$D_i = \sum_{j\in[n_{i,j}]} \gamma_{i,j} \partial^{\mathbf{r}_{i,j}}, \; |\mathbf{r}_{i,j}| \leq s, \; \gamma_{i,j} \in \mathbb{R}, \; \partial^{\mathbf{r}} f(\mathbf{x}) = \frac{\partial^{\sum_{j=1}^d r_j} f(\mathbf{x})}{\partial_{x_1}^{r_1} \cdots \partial_{x_d}^{r_d}}.$$

Blanket assumptions

1. Domain $\mathcal{X} \subseteq \mathbb{R}^d$: open. Kernel $k \in \mathcal{C}^s(\mathcal{X} \times \mathcal{X})$.
2. $K_i \subset \mathcal{X}$: compact, $\forall i$.
3. Bias domain $\mathcal{B} \subseteq \mathbb{R}^Q$: convex.
4. Loss L restricted to $\mathcal{B}$: strictly convex in $\mathbf{b}$.
5. Regularizer Ω: strictly increasing in each of its argument.

Our strenghtened SOC-constrained formulation

$$(\mathbf{f}_{\boldsymbol{\eta}}, \mathbf{b}_{\boldsymbol{\eta}}) = \underset{\mathbf{f} \in (\mathcal{H}_k)^Q,\, \boldsymbol{b} \in \mathcal{B}}{\arg\min} \quad \mathcal{L}(\mathbf{f}, \boldsymbol{b}) \tag{$\mathcal{P}_{\boldsymbol{\eta}}$}$$

$$\text{s.t.}$$

$$\begin{aligned} &(\mathbf{b}_0 - \mathbf{U}\mathbf{b})_i + \eta_i \|(\mathbf{W}\mathbf{f} - \mathbf{f}_0)_i\|_{\mathcal{H}_k} \\ &\leq \min_{m \in [M_i]} D_i(\mathbf{W}\mathbf{f} - \mathbf{f}_0)_i \left(\tilde{\mathbf{x}}_{i,m}\right), \ \forall i \in [I], \end{aligned} \tag{$\mathcal{C}_{\boldsymbol{\eta}}$}$$

where

- $\{\tilde{\mathbf{x}}_{i,m}\}_{m \in [M_i]}$: a δ_i-net of K_i in $\|\cdot\|_{\mathcal{X}}$,
- $\eta_i = \sup_{m \in [M_i], \mathbf{u} \in \mathbb{B}_{\|\cdot\|_{\mathcal{X}}}(\mathbf{0},1)} \|D_{i,\mathbf{x}} k(\tilde{\mathbf{x}}_{i,m}, \cdot) - D_{i,\mathbf{x}} k(\tilde{\mathbf{x}}_{i,m} + \delta_i \mathbf{u}, \cdot)\|_{\mathcal{H}_k}$.

Theorem

- Minimal values: v_{disc} = value of $(\mathcal{P}_{\boldsymbol{\eta}})$ with '$\boldsymbol{\eta} = \mathbf{0}$', $\bar{v} = \mathcal{L}\left(\bar{\mathbf{f}}, \bar{\boldsymbol{b}}\right)$, $v_{\boldsymbol{\eta}} = \mathcal{L}\left(\mathbf{f}_{\boldsymbol{\eta}}, \mathbf{b}_{\boldsymbol{\eta}}\right)$.
- Let $\mathbf{f}_{\boldsymbol{\eta}} = \left(f_{\boldsymbol{\eta},q}\right)_{q\in[Q]}$.

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- Let $\mathbf{f}_{\boldsymbol{\eta}} = \left(f_{\boldsymbol{\eta},q}\right)_{q\in[Q]}$.

Then,

- (i) Tightening: any $(\mathbf{f}, \boldsymbol{b})$ satisfying $(\mathcal{C}_{\boldsymbol{\eta}})$ also satisfies $(\mathcal{C})$, hence

$$v_{\text{disc}} \leq \bar{v} \leq v_{\boldsymbol{\eta}}.$$

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- (ii) Representer theorem: For $\forall q \in [Q]$, $\exists \tilde{a}_{i,0,q}, \tilde{a}_{i,m,q}, a_{n,q} \in \mathbb{R}$ s.t.

$$f_{\boldsymbol{\eta},q} = \sum_{i\in[I]} \left[\tilde{a}_{i,0,q} f_{0,i} + \sum_{m\in[M_i]} \tilde{a}_{i,m,q} D_{i,\mathbf{x}} k\left(\tilde{\mathbf{x}}_{i,m}, \cdot\right) \right] + \sum_{n\in[N]} a_{n,q} k(\mathbf{x}_n, \cdot).$$

Theorem – continued

- (iii) Performance guarantee: if $\mathcal{L}$ is $(\mu_{f_q}, \mu_{\boldsymbol{b}})$-strongly convex w.r.t. $(f_q, \mathbf{b})$ for any $q \in [Q]$, then

$$\|f_{\boldsymbol{\eta},q} - \bar{f}_q\|_{\mathcal{H}_k} \leq \sqrt{\frac{2(v_{\boldsymbol{\eta}} - v_{\text{disc}})}{\mu_{f_q}}}, \quad \|\boldsymbol{b}_{\boldsymbol{\eta}} - \bar{\boldsymbol{b}}\|_2 \leq \sqrt{\frac{2(v_{\boldsymbol{\eta}} - v_{\text{disc}})}{\mu_{\boldsymbol{b}}}}.$$

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If in addition $\mathbf{U}$ is surjective, $\mathcal{B} = \mathbb{R}^Q$, and $\mathcal{L}(\bar{\mathbf{f}}, \cdot)$ is L_b-Lipschitz continuous on $\mathbb{B}_{\|\cdot\|_2}\left(\bar{\mathbf{b}}, c_f \|\boldsymbol{\eta}\|_\infty\right)$ where $c_f = \sqrt{d} \left\| \left(\mathbf{U}^T\mathbf{U}\right)^{-1}\mathbf{U}^T \right\| \max_{i\in[I]} \left\|(\mathbf{W}\bar{\mathbf{f}} - \mathbf{f}_0)_i\right\|_{\mathcal{H}_k}$, then

$$\|f_{\boldsymbol{\eta},q} - \bar{f}_q\|_{\mathcal{H}_k} \leq \sqrt{\frac{2L_b c_f \|\boldsymbol{\eta}\|_\infty}{\mu_{f_q}}}, \|\boldsymbol{b}_{\boldsymbol{\eta}} - \bar{\boldsymbol{b}}\|_2 \leq \sqrt{\frac{2L_b c_f \|\boldsymbol{\eta}\|_\infty}{\mu_{\boldsymbol{b}}}}.$$

Theorem – continued

- (iii) Performance guarantee: if $\mathcal{L}$ is $(\mu_{f_q}, \mu_{\boldsymbol{b}})$-strongly convex w.r.t. $(f_q, \mathbf{b})$ for any $q \in [Q]$, then

$$\|f_{\boldsymbol{\eta},q} - \bar{f}_q\|_{\mathcal{H}_k} \leq \sqrt{\frac{2(v_{\boldsymbol{\eta}} - v_{\text{disc}})}{\mu_{f_q}}}, \quad \|\boldsymbol{b}_{\boldsymbol{\eta}} - \bar{\boldsymbol{b}}\|_2 \leq \sqrt{\frac{2(v_{\boldsymbol{\eta}} - v_{\text{disc}})}{\mu_{\boldsymbol{b}}}}.$$

If in addition $\mathbf{U}$ is surjective, $\mathcal{B} = \mathbb{R}^Q$, and $\mathcal{L}(\bar{\mathbf{f}}, \cdot)$ is L_b-Lipschitz continuous on $\mathbb{B}_{\|\cdot\|_2}\left(\bar{\mathbf{b}}, c_f\|\boldsymbol{\eta}\|_\infty\right)$ where $c_f = \sqrt{d}\left\|\left(\mathbf{U}^T\mathbf{U}\right)^{-1}\mathbf{U}^T\right\| \max_{i\in[I]}\left\|(\mathbf{W}\bar{\mathbf{f}} - \mathbf{f}_0)_i\right\|_{\mathcal{H}_k}$, then

$$\|f_{\boldsymbol{\eta},q} - \bar{f}_q\|_{\mathcal{H}_k} \leq \sqrt{\frac{2L_b c_f\|\boldsymbol{\eta}\|_\infty}{\mu_{f_q}}}, \|\boldsymbol{b}_{\boldsymbol{\eta}} - \bar{\boldsymbol{b}}\|_2 \leq \sqrt{\frac{2L_b c_f\|\boldsymbol{\eta}\|_\infty}{\mu_{\boldsymbol{b}}}}.$$

1st bound: computable. 2nd: Larger $M_i \Rightarrow$ smaller $\delta_i \Rightarrow$ smaller η_i $\Rightarrow$ tighter bound.

Tightening idea

Let $s = 0$, $l = 1$. Recall constraint $(\mathcal{C})$:

$$\left\{(\mathbf{f}, \mathbf{b}) \mid \underbrace{(b_0 - \mathbf{Ub})}_{\beta} \leq \underbrace{\underbrace{(\mathbf{Wf} - f_0)}_{\phi}(\mathbf{x})}_{\langle \phi, k(\mathbf{x}, \cdot) \rangle_{\mathcal{H}_k}}, \quad \forall \mathbf{x} \in K \right\}$$

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$$\Phi(K) := \{k(\mathbf{x}, \cdot) : \mathbf{x} \in K\} \subseteq H^+_{\phi,\beta} := \{g \in \mathcal{H}_k \mid \beta \leq \langle \phi, g \rangle_{\mathcal{H}_k}\}$$

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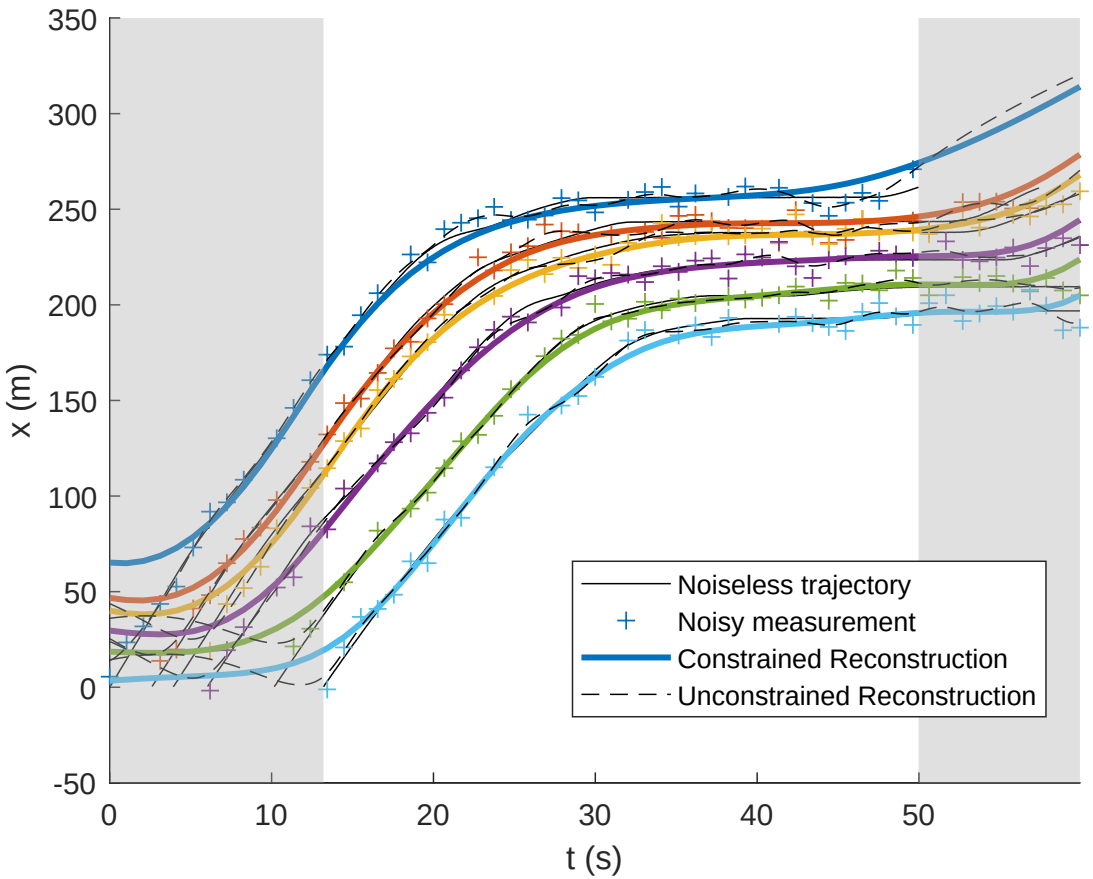
- η is obtained as the minimal radius.

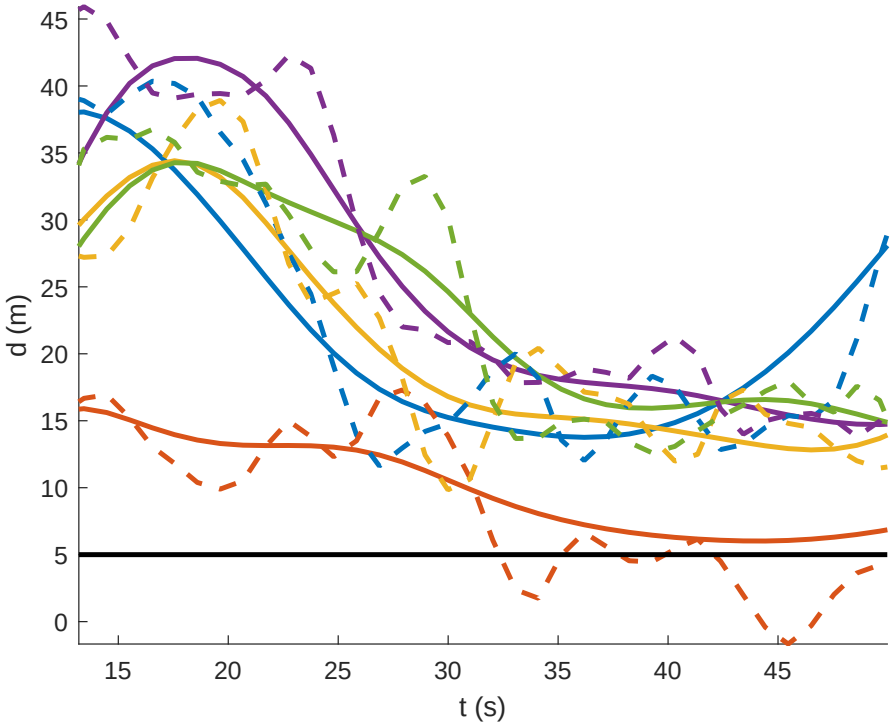
Demo: task-1 = convoy localization with traffic jam

Setting: $q = 6,\ d_{\min} = 5m,\ v_{\min} = 0.$



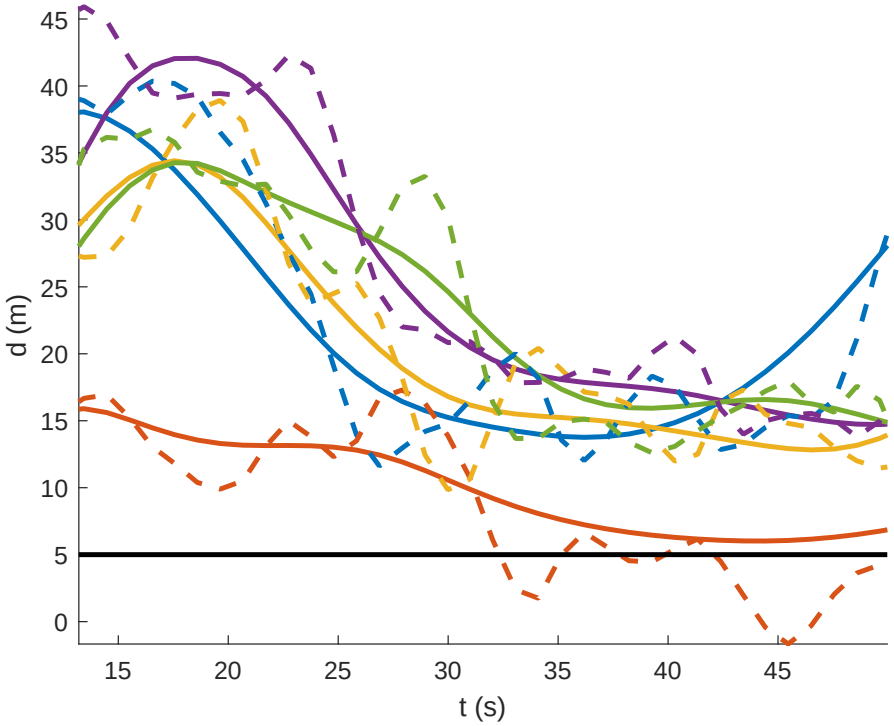
Convoy localization: continued

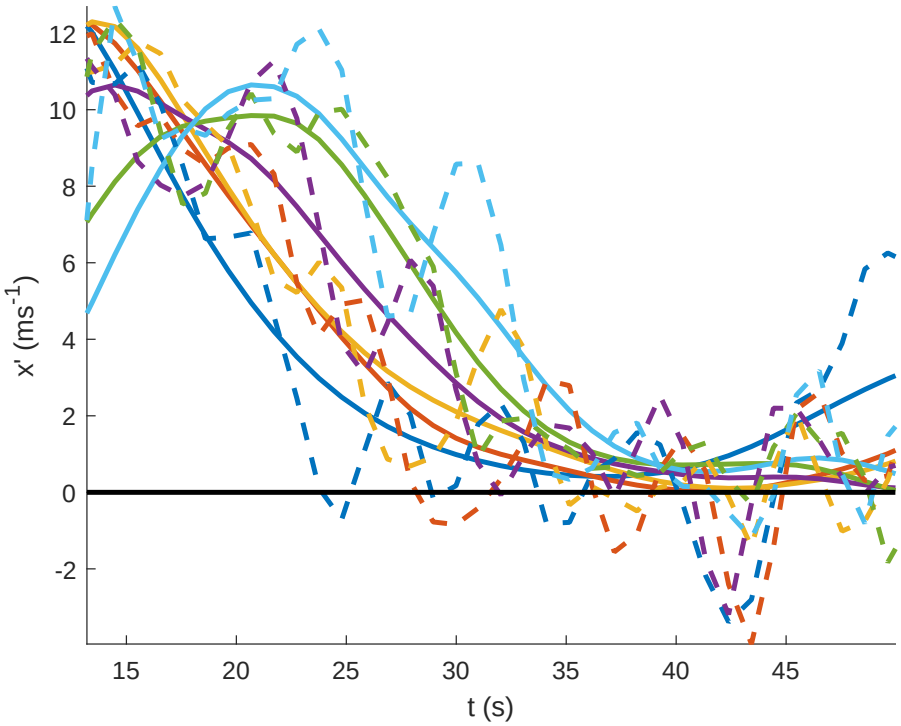
Pairwise distances: $t \mapsto f_q(t) - f_{q+1}(t)$



Convoy localization: continued

Pairwise distances: $t \mapsto f_q(t) - f_{q+1}(t)$ Speed: $t \mapsto f_q'(t)$





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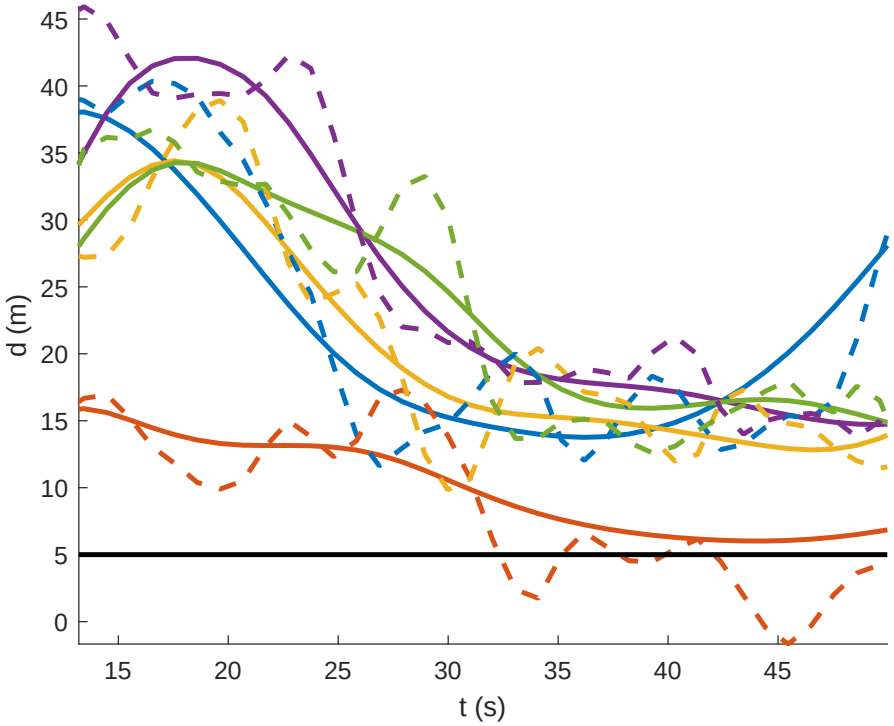
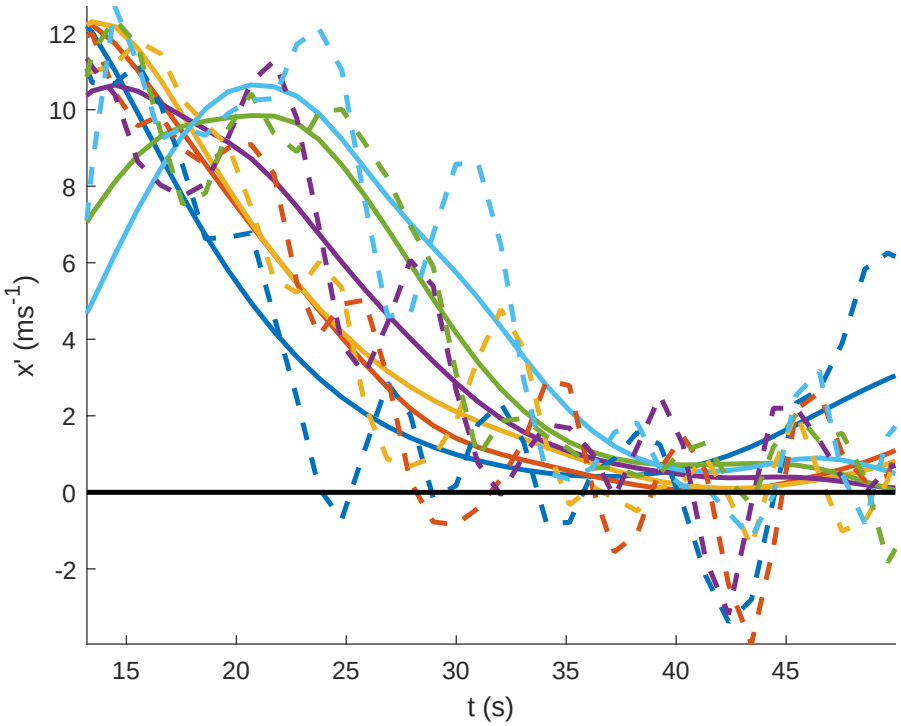


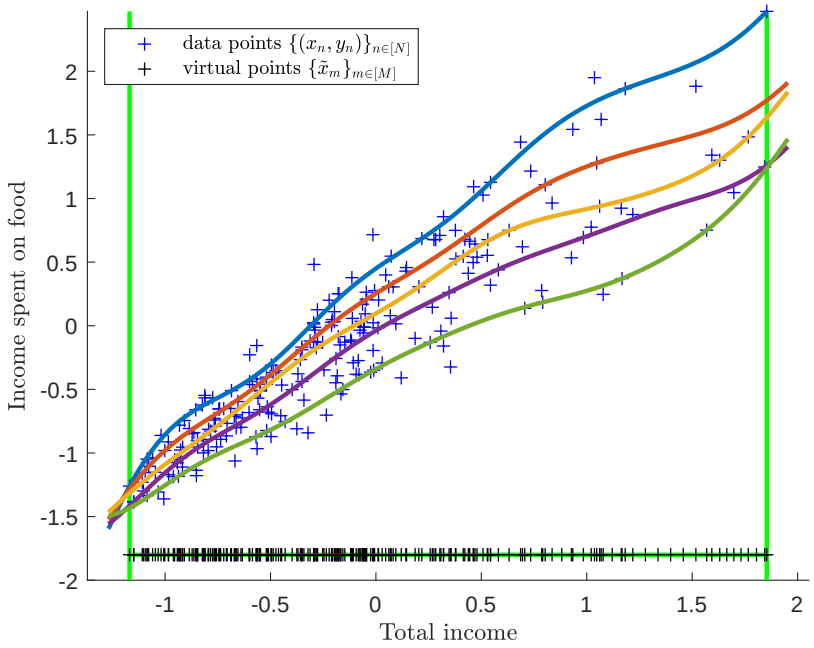


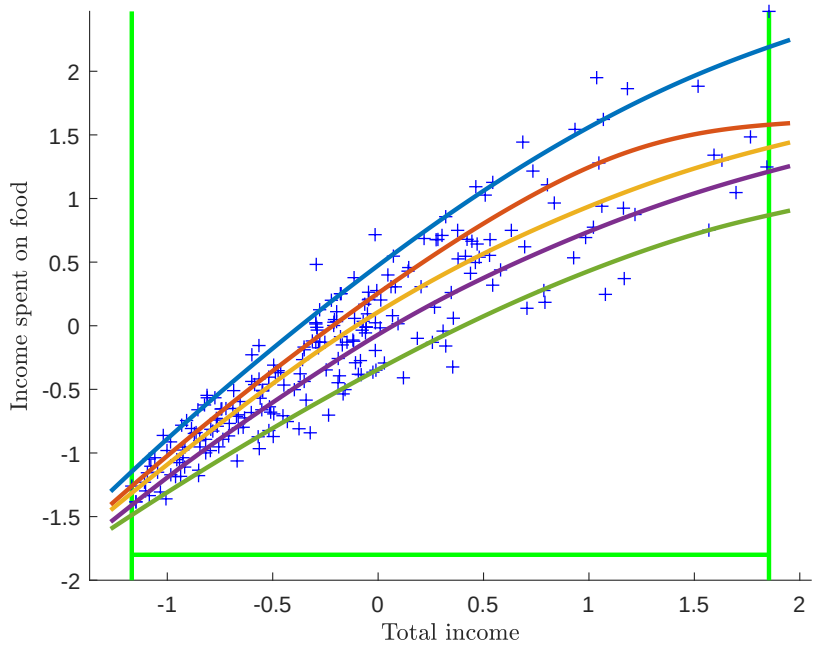
Shape constraints: especially relevant in noisy situations.

Demo: task-2 = joint quantile regression

Economics:

- x: annual household income, y: food expenditure. $d = 1$, $N = 235$.
- Engel's law $\Rightarrow \nearrow$, concave.
- Demo: $\tau_q \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.
- Left: non-crossing, $\nearrow$. Right: non-crossing, $\nearrow$, concave.

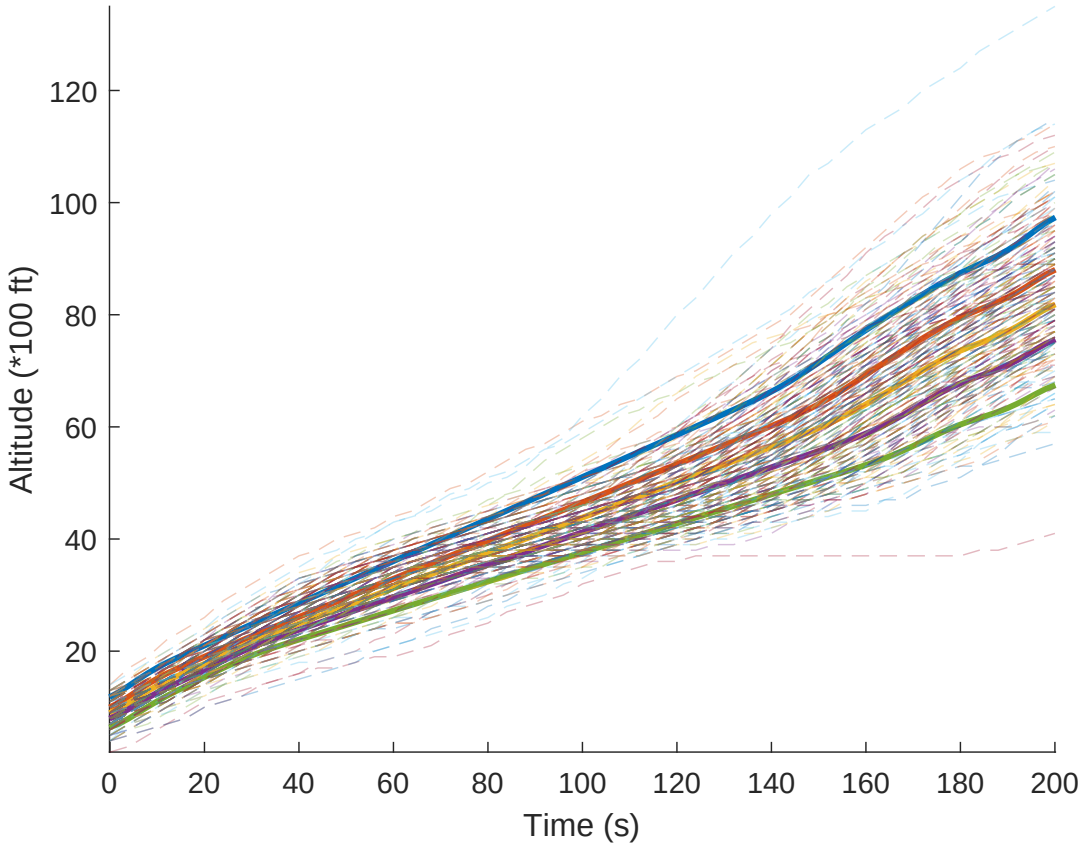




Demo: task-2 = joint quantile regression

Analysis of aircraft trajectories, ENAC:

- y: radar-measured altitude of aircrafts flying between two cities (Paris & Toulouse); x: time. $d = 1$, $N = 15657$.
- Demo: $\tau_q \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.
- Constraint: non-crossing, $\nearrow$ (takeoff).



Summary

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Supplement

- Joint quantile regression on 9 UCI benchmarks.
- Shape constraints: applications areas.
- Partial ordering.

Joint quantile regression: 9 UCI testbeds

- PDCD = Primal-Dual Coordinate Descent [Sangnier et al., 2016].
- 4-5th columns: mean $\pm$ std of 100$\times$value of the pinball loss; smaller is better.

Dataset	d	N	PDCD	SOC
engel	1	235	48 ± 8	53 ± 9
GAGurine	1	314	61 ± 7	65 ± 6
geyser	1	299	105 ± 7	108 ± 3
mcycle	1	133	66 ± 9	62 ± 5
ftcollinssnow	1	93	154 ± 16	148 ± 13
CobarOre	2	38	159 ± 24	151 ± 17
topo	2	52	69 ± 18	62 ± 14
caution	2	100	88 ± 17	98 ± 22
ufc	3	372	81 ± 4	87 ± 6

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- Biology (monotone regression): identify genome interactions [Luss et al., 2012], dose-response studies [Hu et al., 2005].

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- Supply chain models, stochastic multi-period inventory problems, pricing models and game theory: supermodularity [Topkis, 1998, Simchi-Levi et al., 2014].

Partial ordering

$(\mathcal{X}, \leq)$ is a partial ordering if

1. reflexity: $a \leq a$ for $\forall a \in \mathcal{X}$,
2. antisymmetry: $a \leq b$ and $b \leq a$ imply $a = b$, and
3. transitivity: if $a \leq b$ and $b \leq c$ imply $a \leq c$

hold.


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