Minimax-optimal distribution regression

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Joint work with

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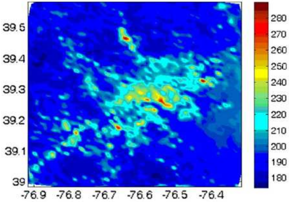
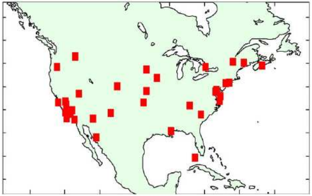
Example: sustainability

- **Goal**: aerosol prediction = air pollution $\rightarrow$ climate.

- Prediction using labelled bags:
  - bag := multi-spectral satellite measurements over an area,
  - label := local aerosol value.

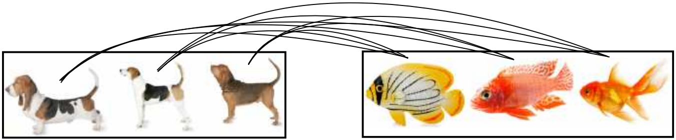




Existing methods

Multi-instance learning:

- [Haussler, 1999, Gärtner et al., 2002] (set kernel):

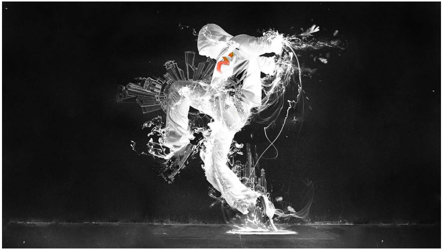


- sensible methods in regression: few,
  1. restrictive technical conditions,
  2. super-high resolution satellite image: would be needed.

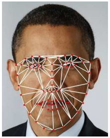
One-page summary

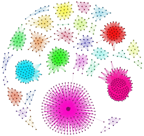
Contributions:

1. Practical: state-of-the-art accuracy (aerosol).
2. Theoretical:
   - General bags: graphs, time series, texts, ...
   - Consistency of set kernel in regression (17-year-old open problem).
   - How many samples/bag?



Objects in the bags





- Examples:
  - time-series modelling: user = set of time-series,
  - computer vision: image = collection of patch vectors,
  - NLP: corpus = bag of documents,
  - network analysis: group of people = bag of friendship graphs, ...

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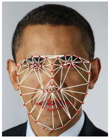

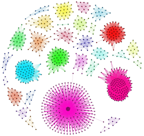

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  - NLP: corpus = bag of documents,
  - network analysis: group of people = bag of friendship graphs, . . .
- Wider context (statistics): point estimation tasks.

Regression on labelled bags

- Given:
  - labelled bags: $\hat{\mathbf{z}} = \left\{\left(\hat{P}_i, y_i\right)\right\}_{i=1}^{\ell}$, $\hat{P}_i$: bag from P_i, $N := |\hat{P}_i|$.
  - test bag: $\hat{P}$.

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- Estimator:

$$f_{\hat{\mathbf{z}}}^{\lambda} = \arg\min_{f \in \mathcal{H}} \frac{1}{\ell} \sum_{i=1}^{\ell} \left[f\underbrace{\left(\mu_{\hat{P}_i}\right)}_{\text{feature of } \hat{P}_i} - y_i \right]^2 + \lambda \|f\|_{\mathcal{H}}^2 .$$

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- Prediction:

$$\hat{y}\left(\hat{P}\right) = \mathbf{g}^T (\mathbf{G} + \ell\lambda\mathbf{I})^{-1}\mathbf{y},$$
$$\mathbf{g} = \left[K\left(\mu_{\hat{P}}, \mu_{\hat{P}_i}\right)\right], \mathbf{G} = \left[K\left(\mu_{\hat{P}_i}, \mu_{\hat{P}_j}\right)\right], \mathbf{y} = [y_i].$$

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Challenges

1. Inner product of distributions: $K\left(\mu_{\hat{P}_i}, \mu_{\hat{P}_j}\right) = ?$
2. How many samples/bag?

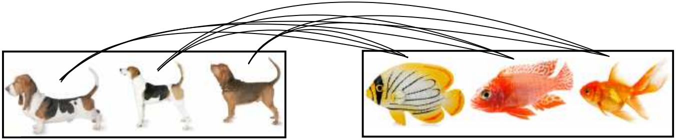
Regression on labelled bags: similarity

Let us define an inner product on distributions [$\tilde{K}(P,Q)$]:

1. Set kernel: $A = \{a_i\}_{i=1}^N$, $B = \{b_j\}_{j=1}^N$.

$$\tilde{K}(A,B) = \frac{1}{N^2}\sum_{i,j=1}^{N} k(a_i,b_j) = \left\langle \underbrace{\frac{1}{N}\sum_{i=1}^{N}\varphi(a_i)}_{\text{feature of bag } A}, \frac{1}{N}\sum_{j=1}^{N}\varphi(b_j) \right\rangle.$$

Remember:



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2. Taking 'limit' [Berlinet and Thomas-Agnan, 2004, Altun and Smola, 2006, Smola et al., 2007]: $a \sim P, b \sim Q$

$$\tilde{K}(P, Q) = \mathbb{E}_{a,b}k(a, b) = \left\langle \underbrace{\mathbb{E}_a\varphi(a)}_{\text{feature of distribution } P =: \mu_P}, \mathbb{E}_b\varphi(b)\right\rangle.$$

Example (Gaussian kernel): $k(\mathbf{a}, \mathbf{b}) = e^{-\|\mathbf{a}-\mathbf{b}\|_2^2/(2\sigma^2)}$.

Regression on labelled bags: baseline

Quality of estimator, baseline:

$$\mathcal{R}(f) = \mathbb{E}_{(\mu_P,y)\sim\rho}[f(\mu_P) - y]^2,$$
$$f_\rho = \text{best regressor}.$$

How many samples/bag to get the accuracy of f_ρ? Possible?

Assume (for a moment): $f_\rho \in \mathcal{H}(K)$.

Our result: how many samples/bag

- Known [Caponnetto and De Vito, 2007]: best/achieved rate

$$\mathcal{R}(f_{\mathbf{z}}^{\lambda}) - \mathcal{R}(f_{\rho}) = \mathcal{O}\left(\ell^{-\frac{bc}{bc+1}}\right),$$

b – size of the input space, c – smoothness of f_{ρ}.

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Our result

- If $2 \leq a$, then $f_{\hat{\mathbf{z}}}^{\lambda}$ attains the best achievable rate.
- In fact, $a = \frac{b(c+1)}{bc+1} < 2$ is enough.
- Consequence: regression with set kernel is consistent.

Aerosol prediction result ($100 \times RMSE$)

We perform on par with the state-of-the-art, hand-engineered method.

- Zhuang Wang, Liang Lan, Slobodan Vucetic. IEEE Transactions on Geoscience and Remote Sensing, 2012: $7.5 - 8.5$ ($\pm 0.1 - 0.6$):
  - hand-crafted features.
- Our prediction accuracy: 7.81 (± 1.64).
  - no expert knowledge.
- Code in ITE: #2 on mloss,

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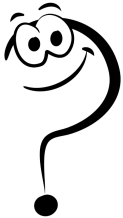
https://bitbucket.org/szzoli/ite/
```

Summary

- Task: regression on bags/distributions.
- Result:
  - minimax optimality, sub-quadratic bag size,
  - specifically: set kernel is consistent.
- Preprint (JMLR, in revision):

`http://arxiv.org/abs/1411.2066`

Thank you for the attention!



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Why can we get consistency/rates? – intuition

- Convergence of the mean embedding:

$$\left\|\mu_P - \mu_{\hat{P}}\right\|_H = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

- Hölder property of K $(0 < L,\ 0 < h \leq 1)$:

$$\left\|K(\cdot, \mu_P) - K(\cdot, \mu_{\hat{P}})\right\|_{\mathcal{H}} \leq L \left\|\mu_P - \mu_{\hat{P}}\right\|_H^h.$$

- $f_{\hat{z}}^{\lambda}$ depends 'nicely' on $K(\mu_{\hat{P}}, \mu_{\hat{Q}}) = \left\langle K(\cdot, \mu_{\hat{P}}), K(\cdot, \mu_{\hat{Q}}) \right\rangle_{\mathcal{H}}$. [39 pages]

Extensions

1. Misspecified setting ($f_\rho \in L^2 \backslash \mathcal{H}$):
   - Consistency: convergence to $\inf_{f \in \mathcal{H}} \|f - f_\rho\|_{L^2}$.
   - Smoothness on f_ρ: computational & statistical tradeoff.

Extensions

2. Vector-valued output:
   - Y: separable Hilbert space $\Rightarrow K(\mu_P, \mu_Q) \in \mathcal{L}(Y)$.
   - Prediction on a test bag $\hat{P}$:

$$\hat{y}(\hat{P}) = \mathbf{g}^T(\mathbf{G} + \ell\lambda\mathbf{I})^{-1}\mathbf{y},$$
$$\mathbf{g} = [K(\mu_{\hat{P}}, \mu_{\hat{P}_i})], \mathbf{G} = [K(\mu_{\hat{P}_i}, \mu_{\hat{P}_j})], \mathbf{y} = [y_i].$$

Specifically: $Y = \mathbb{R} \Rightarrow \mathcal{L}(Y) = \mathbb{R}$; $Y = \mathbb{R}^d \Rightarrow \mathcal{L}(Y) = \mathbb{R}^{d\times d}$.

Other valid similarities

Recall: $\tilde{K}(P,Q) = \langle \mu_P, \mu_Q \rangle$.

$\tilde{K}_G$	$\tilde{K}_e$	$\tilde{K}_C$
$e^{-\frac{\left\Vert \mu_P - \mu_Q \right\Vert^2}{2\theta^2}}$	$e^{-\frac{\left\Vert \mu_P - \mu_Q \right\Vert}{2\theta^2}}$	$\left(1 + \left\Vert \mu_P - \mu_Q \right\Vert^2 / \theta^2\right)^{-1}$

$\tilde{K}_t$	$\tilde{K}_i$
$\left(1 + \left\Vert \mu_P - \mu_Q \right\Vert^\theta\right)^{-1}$	$\left(\left\Vert \mu_P - \mu_Q \right\Vert^2 + \theta^2\right)^{-\frac{1}{2}}$

Functions of $\left\Vert \mu_P - \mu_Q \right\Vert \Rightarrow$ computation: similar to set kernel.

Altun, Y. and Smola, A. (2006).
Unifying divergence minimization and statistical inference via convex duality.
In *Conference on Learning Theory (COLT)*, pages 139–153.

Berlinet, A. and Thomas-Agnan, C. (2004).
Reproducing Kernel Hilbert Spaces in Probability and Statistics.
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Gärtner, T., Flach, P. A., Kowalczyk, A., and Smola, A. (2002).
Multi-instance kernels.
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Haussler, D. (1999).

Convolution kernels on discrete structures.
Technical report, Department of Computer Science, University of California at Santa Cruz.
(http://cbse.soe.ucsc.edu/sites/default/files/convolutions.pdf).

Smola, A., Gretton, A., Song, L., and Schölkopf, B. (2007).
A Hilbert space embedding for distributions.
In *Algorithmic Learning Theory (ALT)*, pages 13–31.