MOM-based robust nonlinear anomaly detection for multispectral and hyperspectral data

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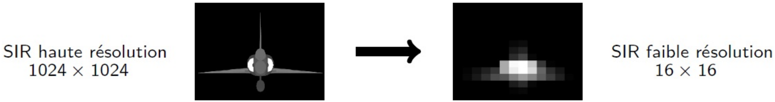
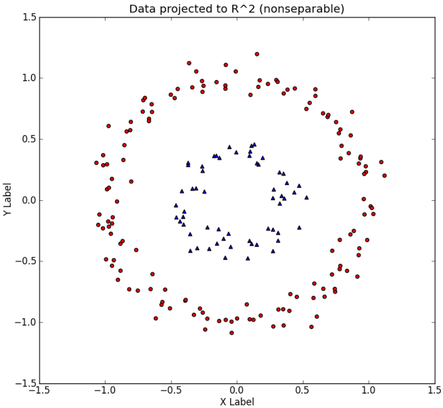
ONERA
THE FRENCH AEROSPACE LAB

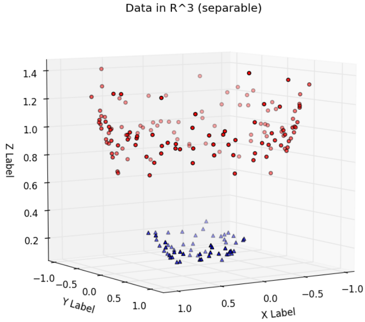
ONERA Plane dataset - Multispectral dataset

- Simulated multispectral **aircraft signatures** by ONERA

- Distance target/sensor ⇒ low resolution







source : reddit.com/r/MachineLearning/

- $\mathcal{X}$ ambient space
- $\mathcal{H}$ is a reproducing kernel Hilbert space (RKHS)
- $\exists! K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that $K(\cdot, x) \in \mathcal{H}$ for all $x \in \mathcal{H}$ and $f(x) = \langle K(\cdot, x), f \rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$, $x \in \mathcal{X}$.
- K is called the reproducing kernel of $\mathcal{H}$ as

$$K(x, y) = \langle K(\cdot, x), K(\cdot, y) \rangle_{\mathcal{H}}$$

  and is symmetric and positive definite.
- $\mathcal{H} = \overline{\text{span}\{K(\cdot, x) : x \in \mathcal{X}\}}$ (linear span of kernel functions)

- **Mean embedding*** of prob. meas. $\mathbb{P}$

$$\mu_{\mathbb{P}} = \int_{\mathcal{X}} K(\cdot, x) \mathrm{d}\mathbb{P}(x) \in \mathcal{H}_K,$$

- $\mu_{\mathbb{P}}$ is well defined
  - ▷ when $\int_{\mathcal{X}} \sqrt{K(x,x)} \mathrm{d}\mathbb{P}(x) < \infty$,
  - ▷ specifically if K is bounded, example: Gaussian kernel.

$\Rightarrow$ representation of a prob. dist. on $\mathcal{X}$ as a point in $\mathcal{H}_K$.

- **Mean reproducing property:**

$$\mathbb{E}_{x \sim \mathbb{P}} f(x) = \langle f, \mu_{\mathbb{P}} \rangle_{\mathcal{H}_K} \quad (\forall f \in \mathcal{H}_K)$$

*Berlinet & Thomas-Agnan, 2004 ; Smola et al., 2007

- **Maximum mean discrepancy** (MMD)*

$$\text{MMD}_K(\mathbb{P}, \mathbb{Q}) = \|\mu_{\mathbb{P}} - \mu_{\mathbb{Q}}\|_{\mathcal{H}_K} .$$

$\Rightarrow$ (semi-)metric on distributions on $\mathcal{X}$.

- **Gretton et al. (2012):** Unbiased estimation

$$\text{MMD}_u(\mathbb{P}, \mathbb{Q}) = \frac{1}{N(N-1)} \sum_{i \neq j = 1}^{N} h(z_i, z_j),$$

where $z_i = (x_i, y_i)$ and
$h(z_i, z_j) = K(x_i, x_j) + K(y_i, y_j) - K(x_i, y_j) - K(x_j, y_i)$.

*Smola et al., 2007 ; Gretton et al., 2012

- Intuitively, MONs* replace the linear operation of expectation with the median of averages taken over non-overlapping blocks of the data, in order to get a robust estimate thanks to the median step.

$$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad \cdots \quad x_{N-2} \quad x_{N-1} \quad x_N$$

$$\mathbb{P}_{B_1} := \frac{1}{|B_1|}(\delta_{x_1} + \delta_{x_2} + \delta_{x_3}) \qquad \mathbb{P}_{B_2} \qquad \mathbb{P}_{B_Q}$$

- Median Of meaN (MON)

$$\mathrm{MON}_Q[f] := \underset{q \in [Q]}{\mathrm{med}} \left\{ \mathbb{P}_{B_q} f \right\} = \underset{q \in [Q]}{\mathrm{med}} \left\{ \left\langle f, \mu_{B_q} \right\rangle_{\mathcal{H}_K} \right\}.$$

*Jerrum et al., 1986 ; Lugosi & Mendelson, 2017

- Robust estimation of the mean embedding based on MON: $\hat{\mu}_{\mathbb{P},Q}$

$$\hat{\mu}_{\mathbb{P},Q} = \hat{\mu}_{\mathbb{P},Q}(x_{1:N}) \in \operatorname*{argmin}_{f \in \mathcal{H}_K} \sup_{g \in \mathcal{H}_K} J(f,g),$$

where $J(f,g) = \mathrm{MON}_Q\left[\|f - K(\cdot,x)\|_K^2 - \|g - K(\cdot,x)\|_K^2\right]$.

- Robust estimation of the MMD based on MON: $\widehat{\mathrm{MMD}}_Q(\mathbb{P},\mathbb{Q})$:

$$\widehat{\mathrm{MMD}}_Q(\mathbb{P},\mathbb{Q}) = \sup_{f \in \mathcal{B}_K} \operatorname*{med}_{q \in [Q]} \left\{ \left\langle f, \mu_{B_q,\mathbb{P}} - \mu_{B_q,\mathbb{Q}} \right\rangle_K \right\}.$$

where $\mathcal{B}_K = \{f \in \mathcal{H}_K : \|f\|_K \leq 1\}$.

- **Mean embedding:**
  - N_c elements $(x_{n_j})_{j=1}^{N_c}$ of the sample are arbitrarily corrupted.
- **MMD:**
  - $\{(x_{n_j}, y_{n_j})\}_{j=1}^{N_c}$ can be contaminated.
- The number of corrupted samples can be (almost) half of the number of blocks: $\exists \delta \in (0, 1/2]$ such that $N_c \leq Q(1/2 - \delta)$.
- $\delta = \frac{1}{2} \Rightarrow$ no contamination.

Theorem (Consistency and outlier robustness)

Assume $\Sigma_{\mathbb{P}}, \Sigma_{\mathbb{Q}} \in \mathcal{L}_1(\mathcal{H}_K)$. *Let* $c_1 = 2\left(1+\sqrt{2}\right)$. *Then with prob. at least* $1 - e^{-\frac{Q\delta^2}{18}}$

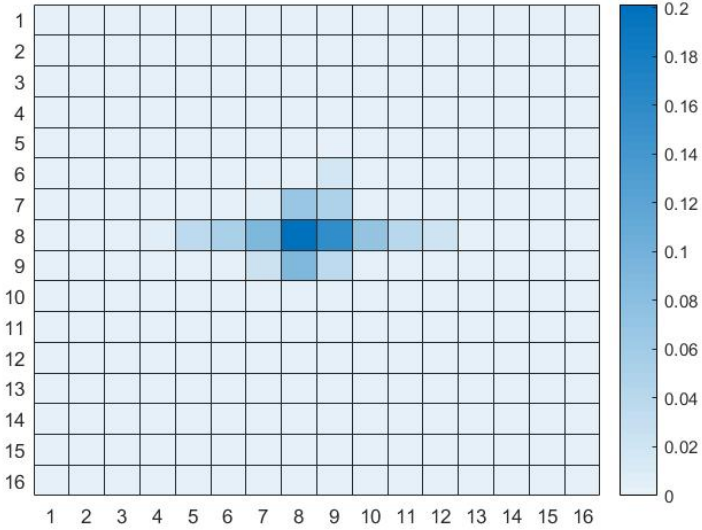
$$\left\|\hat{\mu}_{\mathbb{P},Q} - \mu_{\mathbb{P}}\right\|_K \leq c_1 \max\left(\sqrt{\frac{3\left\|\Sigma_{\mathbb{P}}\right\| Q}{\delta N}}, \frac{12}{\delta}\sqrt{\frac{\mathrm{Tr}\left(\Sigma_{\mathbb{P}}\right)}{N}}\right)$$

and

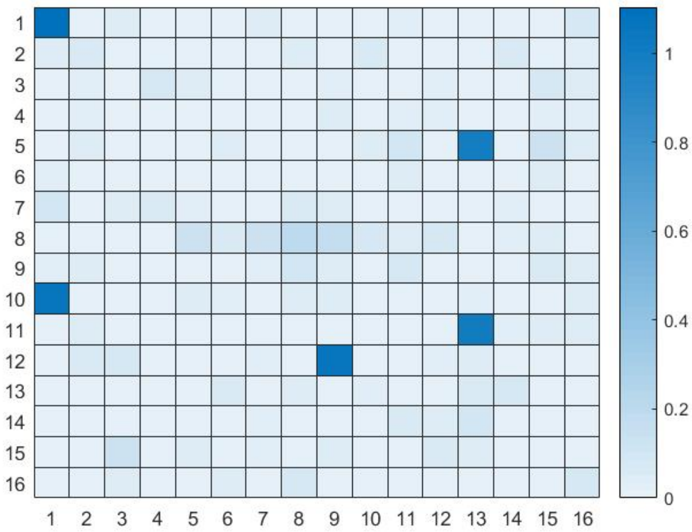
$$\left|\widehat{MMD}_Q(\mathbb{P},\mathbb{Q}) - MMD_Q(\mathbb{P},\mathbb{Q})\right| \leq$$

$$\leq 2\max\left(\sqrt{\frac{3\left(\left\|\Sigma_{\mathbb{P}}\right\| + \left\|\Sigma_{\mathbb{Q}}\right\|\right) Q}{\delta N}} \frac{12}{\delta}\sqrt{\frac{\mathrm{Tr}\left(\Sigma_{\mathbb{P}}\right) + \mathrm{Tr}\left(\Sigma_{\mathbb{Q}}\right)}{N}}\right).$$

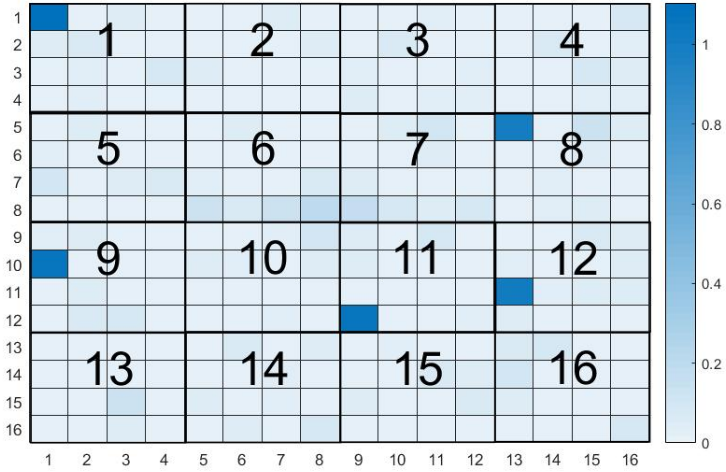
Pixel patches, plane in the middle

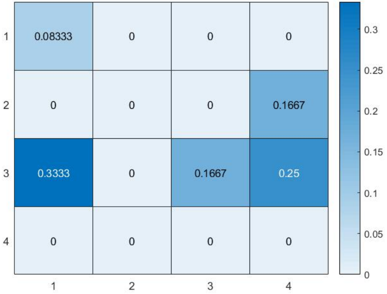


Plane + noise + contamination

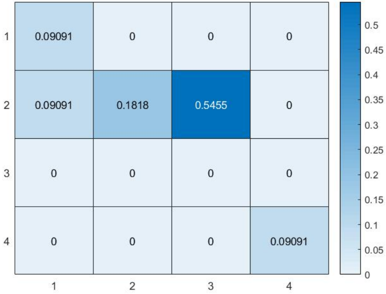


Approximation of the null distribution





Anomaly detection using MMD_u



Anomaly detection using MONK estimator

Conclusion

- New outlier-robust mean embedding and MMD estimators.
- Obtained estimators
  - obey optimal sub-Gaussian deviation bounds
  - are robust to contamination

Lerasle, M., Szabó, Z., Lecué, G., Massiot, G., & Moulines, E. (2018). MONK – Outlier-Robust Mean Embedding Estimation by Median-of-Means. *arXiv preprint arXiv:1802.04784.*

References I

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- Mean embedding

Muandet, K., Fukumizu, K., Sriperumbudur, B., & Schölkopf, B. (2017). Kernel mean embedding of distributions: A review and beyond. *Foundations and Trends in Machine Learning, 10*(1-2), 1-141.

- MOM / MON

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