Kernel Regression for Vehicle Trajectory Reconstruction under Speed and Inter-vehicular Distance Constraints

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This is an introductory paper to a new kernel method to deal with an infinite number of constraints. For more theory, see

Hard Shape-Constrained Kernel Machines, PCAF and Zoltán Szabó
https://arxiv.org/abs/2005.12636

Formulation of practical problem

We want to reconstruct trajectories $t \mapsto f(t)$ based on noisy position measurements $\{x_n\}_{n \leq N}$ recorded at times $\{t_n\}_{n \leq N}$.

- if the noise is small and the sampling rate is large $\Rightarrow$ “easy“
  $\hookrightarrow$ as with camera-collected data
- if the noise is large and the sampling rate is small $\Rightarrow$ “hard“
  $\hookrightarrow$ as with GPS-collected data

One can incorporate some side information, a.k.a priors: times of lane changes, speed limits, non-overtaking while on same lane,. . .

This means using some contextual information from the road description and from the other drivers.

With GPS data, priors are ubiquitous (e.g. map-matching)!

Formulation of optimization problem (through splines)

We want to smooth the noise and get continuous functions that can be further analyzed (e.g. derivated, compared,...).

Assume we have already projected the car to its correct lane, i.e. $x_n \in \mathbb{R}$ is the position on the lane. We want to have $f(t_n) \approx x_n$.

$$\text{Let } \mathcal{T} \subset \mathbb{R},\ \mathcal{F} := W_2^m(\mathcal{T}) = \{f : \mathcal{T} \to \mathbb{R} \mid \int_{\mathcal{T}} |f^{(m)}(t)|^2 \mathrm{d}t < \infty\}$$

For these *splines* in the Sobolev space $\mathcal{F}$, the problem writes as

$$\min_{f \in \mathcal{F}} \Big[\underbrace{\frac{1}{N} \sum_{n \in [N]} |x_n - f(t_n)|^2}_{\text{approximation error}} + \lambda \underbrace{\int_{\mathcal{T}} |f^{(m)}(t)|^2 \mathrm{d}t}_{\text{smoothing term}} \Big].$$

A few words on splines of Sobolev spaces

The space of splines can be decomposed as $W_2^m(\mathcal{T}) = \mathcal{F}_1 \oplus \mathcal{F}_2$

$$\mathcal{F}_1 = \operatorname{span}(1, t, \ldots, \frac{t^{m-1}}{(m-1)!}),$$

$$\mathcal{F}_2 = \left\{f \in W_2^m(\mathcal{T}) \mid f^{(j)}(0) = 0 \ (\forall j \in \{0, \ldots, m\})\right\},$$

$$\|f\|_{\mathcal{F}}^2 = \sum_{j=0}^{m} |f^{(j)}(0)|^2 + \int_{\mathcal{T}} \left|f^{(m)}(t)\right|^2 \mathrm{d}t = \|f_1\|_{\mathcal{F}_1}^2 + \|f_2\|_{\mathcal{F}_2}^2$$

See e.g. (Berlinet, Thomas-Agnan, 2003 [2]).
The projection onto $\mathcal{F}_1$ is not penalized, so

$$\min_{f \in \mathcal{F}} \left[\frac{1}{N} \sum_{n \in [N]} |x_n - f(t_n)|^2 + \lambda \|f_2\|_{\mathcal{F}_2}^2\right].$$

Formulation of optimization problem (through kernels)

Keep the structure $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ but take

- $\mathcal{F}_1$ to be the space of constant functions $\{t \mapsto b \in \mathbb{R}\}$
- $\mathcal{F}_2 = \mathcal{F}_k$ to be a reproducing kernel Hilbert space (RKHS)

for a positive definite kernel $k : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ (defined on next slide).

This leads to kernel ridge regression (a.k.a. abstract spline)

$$\min_{b\in\mathbb{R},\, f\in\mathcal{F}_k} \left[\frac{1}{N} \sum_{n\in[N]} \left| x_n - (b + f(t_n)) \right|^2 + \lambda \|f\|_{\mathcal{F}_k}^2 \right]$$

Reproducing kernel Hilbert spaces (RKHS) at a glance (1)

A RKHS $(\mathcal{F}_k(\mathcal{T}), \langle \cdot, \cdot \rangle_{\mathcal{F}_k})$ is a Hilbert space of real-valued functions over a set $\mathcal{T}$ if one of the following is satisfied (Aronszajn, 1950 [1])

$$\exists\, k : \mathcal{T} \times \mathcal{T} \to \mathbb{R} \text{ s.t. } k_t(\cdot) = k(t, \cdot) \in \mathcal{F}_k(\mathcal{T}) \text{ and } f(t) = \langle f, k_t \rangle_{\mathcal{F}_k}$$

k is s.t. $\mathbf{G} = [k(t_i, t_j)]_{i,j=1}^{M} \succcurlyeq 0$, i.e. (semi)positive definite

- $\forall M, \forall (a_i, t_i) \in (\mathbb{R} \times \mathcal{T})^M, \sum_{i,j \leq M} a_i a_j k(t_i, t_j) = \mathbf{a}^\top \mathbf{G} \mathbf{a} \geq 0$
- $\forall t, s \in \mathcal{T},\ k(t, s) = k(s, t)$

- There is a one-to-one correspondence between kernels k and RKHSs $(\mathcal{F}_k(\mathcal{T}), \langle \cdot, \cdot \rangle_{\mathcal{F}_k})$. Changing $\mathcal{T}$ or $\langle \cdot, \cdot \rangle_{\mathcal{F}_k}$ changes the kernel k.
- if $\mathcal{T}$ is an interval, $k \in \mathcal{C}^{m,m}(\mathcal{T}, \mathcal{T})$, D a differential operator of order at most m, then kernel trick for derivatives holds

$$D_t k(t, \cdot) \in \mathcal{F}_k(\mathcal{T}) \quad ; \quad Df(t) = \langle f(\cdot), D_t k(t, \cdot) \rangle_{\mathcal{F}_k}$$

Reproducing kernel Hilbert spaces (RKHS) at a glance (2)

Sobolev spaces $W_2^m(\mathcal{T})$ are RKHSs. For $\mathcal{T} = \mathbb{R}$, their (Matérn) kernels are well known. For a bandwidth $\sigma > 0$, examples are

$$k_{\text{Gauss}}(t,s) = \exp\left(-\|s-t\|^2_{\mathbb{R}^d}/(2\sigma^2)\right)$$
$$k_{\text{Matérn},3/2}(t,s) = (1+\sqrt{3}|t-s|/\sigma)e^{-\sqrt{3}|t-s|/\sigma}$$

Calculus in RKHSs

If $f(\cdot) = \sum_{i\in[I]} a_i k(\cdot, t_i)$ and $g(\cdot) = \sum_{j\in[J]} a'_j k(\cdot, s_j)$, then

$$\langle f(\cdot), g(\cdot)\rangle_{\mathcal{F}_k} = \sum_{i\in[I]}\sum_{j\in[J]} a_i a'_j k(t_i, s_j)$$

$$\|f(\cdot)\|^2_{\mathcal{F}_k} = \sum_{i\in[I]}\sum_{j\in[J]} a_i a_j k(t_i, t_j) = \mathbf{a}^\top \mathbf{G}\mathbf{a} \qquad \mathbf{G} = [k(t_i, t_j)]_{i,j}$$

Adding a (lower) speed limit

Recall that $b + f(t)$ is the position of the car on its lane at time t

"Cars go forward on a highway"

$$\min_{b \in \mathbb{R}, f \in \mathcal{F}_k} \left[\frac{1}{N} \sum_{n \in [N]} |x_n - (b + f(t_n))|^2 + \lambda \|f\|^2_{\mathcal{F}_k} \right]$$

$$\text{s.t.}$$

$$v_{\min} \leq f'(t), \quad \forall t \in \mathcal{T}.$$

We take $v_{\min} = 0\, ms^{-1}$. Similarly, consider $f'(t) \leq v_{\max}$. Problem:

For $\mathcal{T} = [0, T]$, we have an infinite number of constraints!

Discretize constraint at $\bigcup\{t_i\} \subset \mathcal{T}$? No guarantees in between...

Adding a non-crossing constraint

Recall that $b_q + f_q(t)$ is the position of car q on its lane at time t

"Cars keep a safety distance while on same lane"

$$\min_{\substack{f_1,\dots,f_Q \in \mathcal{F}_k,\\ b_1,\dots,b_Q \in \mathbb{R}}} \frac{1}{Q}\sum_{q=1}^{Q}\left[\left(\frac{1}{N_q}\sum_{n=1}^{N_q}\left|x_{q,n} - (b_q + f_q(t_{q,n}))\right|^2\right) + \lambda \|f_q\|^2_{\mathcal{F}_k}\right]$$

s.t.

$$d_{\min} + b_{q+1} + f_{q+1}(t) \leq b_q + f_q(t), \forall q \in [Q-1],\ t \in \mathcal{T},$$
$$v_{\min} \leq f_q'(t), \qquad \forall q \in [Q],\ t \in \mathcal{T}.$$

Let $d_{\min} = 5\,m$ for the Q cars, $\{t_m\}_{m\in[M]} := \cup_{q\in[Q],n\in[N_q]}\{t_{q,n}\}$.

Assume that $f_q(\cdot) = \sum_{m\in[M]} a_{q,m} k_{t_m}(\cdot)$
$\hookrightarrow$ can be formalized with representer theorem if no speed limits.

Turning an infinite number of constraints into a finite one

Take $k_{\text{Matérn},3/2}(t,s) = (1+\sqrt{3}|t-s|/\sigma)e^{-\sqrt{3}|t-s|/\sigma}$

$$\text{“}v_{\min} \le f_q'(t),\ \forall t \in \mathcal{T}\text{“} \Leftrightarrow \text{“}v_{\min} \le \inf_{t\in\mathcal{T}} f_q'(t)\text{“}$$

As $\sup_{t\in\mathcal{T}} f_q'(t) = \sup_{t\in\mathcal{T}} \left\langle f_q', k(t,\cdot)\right\rangle_{\mathcal{F}_k} \le \underbrace{\sup_{t\in\mathcal{T}} \sqrt{k(t,t)}}_{=1} \|f_q'(\cdot)\|_{\mathcal{F}_k}$,

the following constraints are stronger than “$v_{\min} \le \inf_{t\in\mathcal{T}} f_q'(t)$“

$$\|f_q'(\cdot)\|_{\mathcal{F}_k} \le -v_{\min} \text{ (too conservative for } v_{\min} \ge 0)$$

or, for some arbitrary negative functions $g_m(\cdot) \in \mathcal{F}_k$ and $\beta_{q,m} \ge 0$,

$$\|f_q'(\cdot) + \sum_{m\in[M]} \beta_{q,m} g_m(\cdot)\|_{\mathcal{F}_k} \le -v_{\min} - \sum_{m\in[M]} \sup_{t\in\mathcal{T}}[\beta_{q,m} g_m(t)]$$

or, cover $\mathcal{T}$ by $\bigcup_m [t_m - \delta_m, t_m + \delta_m]$, set $u_m = \sup_{|t-t_m|\le\delta_m} g_m(t)$

$$\|f_q'(\cdot) + \beta_{q,m} g_m(\cdot)\|_{\mathcal{F}_k} \le -v_{\min} - \beta_{q,m} u_m,\ (\forall m \in [M])$$

Expression of convex problem in finite dimensions

Use reproducing formula for $f_q(\cdot) = \sum_{m \in [M]} a_{q,m} k_{t_m}(\cdot)$. Set $\kappa = 1$

$$\min_{\substack{\boldsymbol{a}_1,\dots,\boldsymbol{a}_Q \in \mathbb{R}^M,\\ b_1,\dots,b_Q \in \mathbb{R},\\ \boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_Q \in \mathbb{R}^N_+,\\ \boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_Q \in \mathbb{R}^N_+}} \frac{1}{Q}\sum_{q=1}^{Q} \frac{1}{N_q}\Big[\boldsymbol{a}_q^T(\mathbf{G}_M \boldsymbol{\Pi}_q^T \boldsymbol{\Pi}_q \mathbf{G}_M + \lambda N_q \mathbf{G}_M)\boldsymbol{a}_q$$

$$+ N_q b_q^2 + 2\left(b_q \mathbf{1}_{N_q} - \boldsymbol{x}_q\right)^T \boldsymbol{\Pi}_q \mathbf{G}_M \boldsymbol{a}_q - 2 b_q \mathbf{1}_{N_q}^T \boldsymbol{x}_q\Big]$$

s.t.

$$\kappa \left\| \mathbf{G}_0^{1/2}[\boldsymbol{a}_q - \boldsymbol{a}_{q+1}; \boldsymbol{\alpha}_q] \right\|_2 \le b_q - b_{q+1} - d_{\min} - \boldsymbol{\alpha}_q^T \boldsymbol{u},\ \forall q \in [Q-1],$$

$$\kappa \left\| \mathbf{G}_D^{1/2}[\boldsymbol{a}_q; \boldsymbol{\beta}_q] \right\|_2 \le -v_{\min} - \boldsymbol{\beta}_q^T \boldsymbol{u},\ \forall q \in [Q],$$

(see article for notations) It is a convex problem $\rightarrow$ CVXGEN [4]
These are “second-order cone“ constraints with quadratic objective.

Caracteristics of MoCoPo dataset

Collected by IFSTTAR on highway near Grenoble (mocopo.ifsttar.fr)

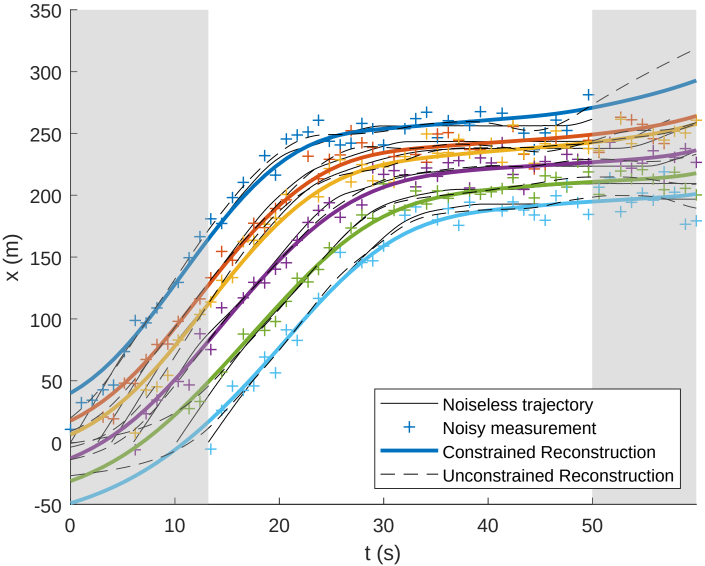
Identification of vehicles from videos taken from helicopter

See (Buisson et al., 2016 [3]) for a reconstruction study through splines on the ‘raw‘ data (small noise, large sampling rate 25 Hz)

We select six vehicles on the same lane reaching a traffic jam.

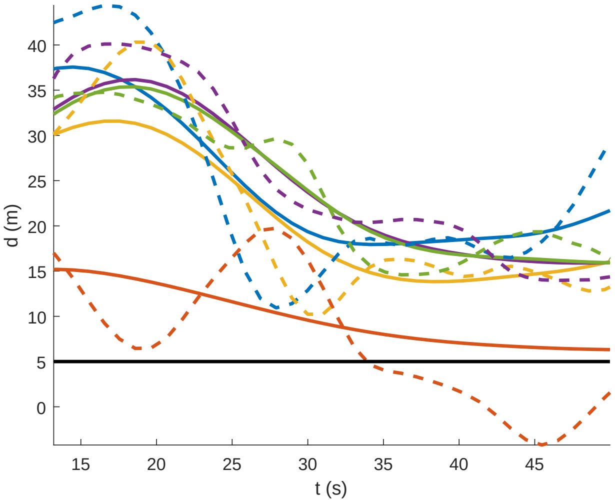
We add large noise (Gaussian with $\sigma_{\text{noise}} > 5\text{m}$) and subsample to 1 Hz to simulate GPS data. We then mask 20% of the measurements (missing data) resulting in 350 data points.

Reconstruction of trajectories near traffic jam



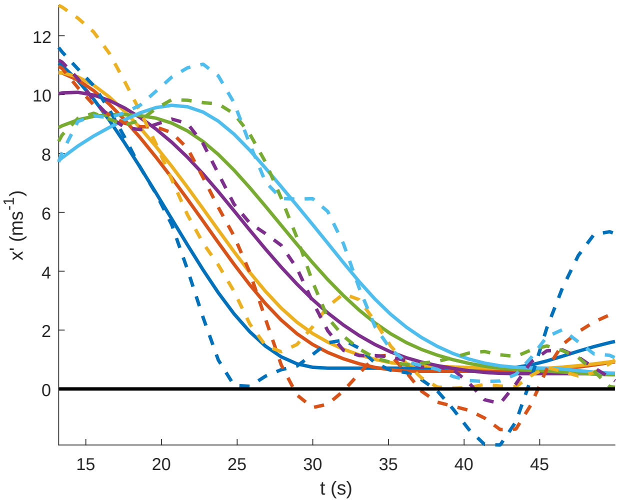
Reconstruction of trajectories (inter-distance ≥ 5)

solid line = constrained, dashed = unconstrained



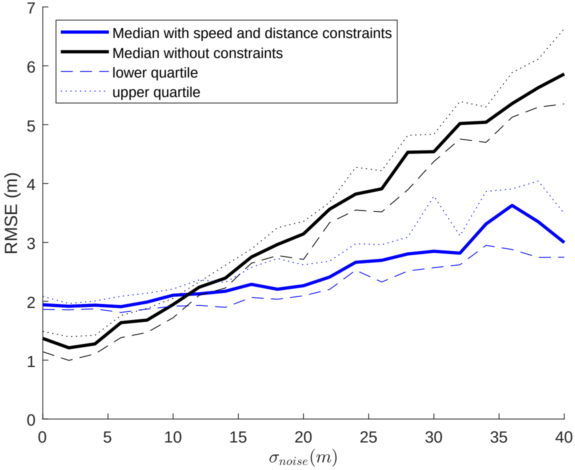
Reconstruction of trajectories (vehicle speeds ≥ 0)

solid line = constrained, dashed = unconstrained



Average reconstruction performance when varying σ_{noise}

Root-mean-square error (RMSE) w.r.t. the ground truth trajectories at the original 25 Hz frequency (repeated 40 times).



Conclusion

Through kernel methods we performed a reconstruction and

- translated priors on behaviors (speed limits, non-overtaking) to convex shape constraints,
- turned an infinite number of constraints into a finite one,
- applied it to real data on a challenging traffic jam.

Thank you for your attention!

Looking for details? See article (and below)! And most of all:

Use kernels!

Hard Shape-Constrained Kernel Machines, PCAF and Zoltán Szabó
`https://arxiv.org/abs/2005.12636`

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Visit `pcaubin.github.io`
for more on kernel methods and control theory

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